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PARTIALLY ORDERED AND RELATIONAL VALUED FUZZY RELATIONS II

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Abstract

Fuzzy relations (partially ordered and relational valued ones) are investigated from the point of view of level relations. A kind of a fuzzy completion of a collection of characteristic functions is introduced and applied on fuzzy relations. In that way, some important notions (composition, equivalence and order induced by a quasiorder etc.) are introduced and their properties investigated.

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1. Introduction

Partially ordered fuzzy sets (mappings from a set to a poset) are introduced in [2], and relational valued ones (codomain of which is a relational system) in [3]. In the paper [6], partially ordered and relational valued relations are introduced and investigated. Because of the absence of the lattice operations in a poset and in a relational system, special relations (similarity, order, quasiorder etc.) are defined by means of level-relations. In addition, all

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fuzzy relations have the property of unique representation by the collection of levels.

In this paper, the investigation of special partially ordered and relational valued relations is continued, again by means of level relations. However, some important notions (composition, equivalence and order induced by a quasiorder etc.) could not be introduced in that way, since the corresponding levels fail to satisfy the necessary properties for the synthesis. To overcome this difficulty, we have introduced a particular completion of a collection of characteristic functions on a set. Namely, in the present article we prove (explicitely describing the algorithm) that for every such collection, there is a partially ordered (or a relational valued) fuzzy set for which these characteristic functions are level sets. The procedure is applied on fuzzy relations, and the above mentioned notions are introduced. Properties of these relational constructions are investigated.

2. Preliminaries: L-valued relations

The following definitions and properties of lattice valued relations are mainly from [1,6]. We shall use them as a motivation for our approach to partially ordered and relational valued relations.

If $(L, \wedge, \vee)$ is a complete lattice (in the following denoted by L) with the bottom (0) and the top (1) element and X a nonempty set, then any mapping $\bar{S} : X^2 \to L$ is an **L-valued (lattice valued) relation** on X.

For $p \in L$, a **p-cut (level-relation)** of $\bar{S}$ is a mapping $\bar{S}_p : X^2 \to \{0, 1\}$, such that

$$\bar{S}_p(x,y) = 1 \text{ iff } \bar{S}(x,y) \geq p.$$

A p-cut $\bar{S}_p$ of $\bar{S}$ is obviously a characteristic function of an (ordinary) relation S_p on X, for which we use the same name - a **p-cut** of $\bar{S}$. Namely, for $p \in L$,

$$(x,y) \in S_p \text{ iff } \bar{S}_p(x,y) = 1 \text{ iff } \bar{S}(x,y) \geq p.$$

We shall denote by $\bar{S}_L$ the family of p-cuts for $\bar{S}$:

$$\bar{S}_L = \{\bar{S}_p \mid p \in L\}.$$

The following formula is said to give the **synthesis** of $\bar{S}$: For $x, y \in X$

$$\bar{S}(x,y) = \bigvee_{p \in L} p \circ \bar{S}_p(x,y), \tag{1}$$

where $\bigvee$ is the supremum in L, and $\circ$ is defined with: $p \circ 0 = 0$, $p \circ 1 = p$.

An L-valued relation $\bar{S}$ on X is a **similarity relation (fuzzy equivalence)** on X if it is

reflexive: $\bar{S}(x,x) = 1$, for every $x \in X$ $(1 \in L)$;

symmetric: $\bar{S}(x,y) = \bar{S}(y,x)$, for all $x, y \in X$;

transitive: $\bar{S}(x,y) \wedge \bar{S}(y,z) \leq \bar{S}(x,z)$, for all $x, y, z \in X$.

If $\bar{S}$ is reflexive and transitive, then it is an L-**valued quasi-ordering** relation on X.

$\bar{S}$ is an L-**valued ordering** on X if it is reflexive, transitive and

antisymmetric:$\bar{S}(x,y) \wedge \bar{S}(y,x) = 0$, for all $x, y \in X$, $x \neq y$, $(0 \in L)$.

If the lattice L is a two-element chain, then ordinary relations satisfying well known properties with the same names are compatible with the above definitions.

If $\bar{S}$ is an L-valued similarity relation on X and $a \in X$, then the mapping $[a]_{\bar{S}} : X \to L$, defined with

$$[a]_{\bar{S}}(x) := \bar{S}(a,x), \; x \in X,$$

is an (L-**valued**) $\bar{S}$-**equivalence class** of a. The collection $\prod_{\bar{S}}$ of all $\bar{S}$-equivalence classes, is an L-**valued partition** of X, induced by $\bar{S}$:

$$\prod\nolimits_{\bar{S}} := \{[a]_{\bar{S}} \mid a \in X\}.$$

If a relation (fuzzy or ordinary) or every relation in a collection satisfies some of the above properties which are not explicitely listed, we shall say that this relation or all the relations in a collection have some **special properties**.

Now we shall describe connections between L-valued relations having special properties and their level-relations. The proofs can be found in the papers listed in References.

Proposition 1. *If an L-valued relation $\bar{S}$ on X has some special properties, then all the p-cuts of that relation are (ordinary) relations on S, satisfying the same properties (i.e. if $\bar{S}$ is reflexive, then for every $p \in L$, S_p is an ordinary reflexive relation on X etc.).*

Remark. *If $\bar{S}$ is antisymmetric, then $S_0 = X^2$.* □

The next proposition is in a sense converse of Proposition 1. It deals with the following problem: which conditions should satisfy a collection of ordinary relations on a set X, all having some special properties, in order to represent the collection of p-cuts for an L-valued relation on X, having same properties?

Proposition 2. *Let $\mathcal{F} = \{\rho_i \mid i \in I\}$ be a family of relations on X all having some special properties, and which satisfy:*

(*i*) *the intersection of any subset of $\mathcal{F}$ belongs to $\mathcal{F}$;*

(*ii*) $X^2 \in \mathcal{F}$.

Let also L be a lattice antiisomorphic with $(\mathcal{F}, \subseteq)$. Define the relation $\bar{S} : X^2 \to L$, so that for $x, y \in X$

$$\bar{S}(x,y) := \bigvee(\rho_i \mid (x,y) \in \rho_i),$$

where $\bigvee$ is the join in L (intersection in F) and the corresponding elements in F and in L are denoted identically. Now, $\bar{S}$ is an L-valued relation on X which has the same special properties as all the relations in $\mathcal{F}$, F is a family of its p-cuts, moreover, $S_{\rho_i} = \rho_i$, for every $i \in I$. □

It is known that on the collection of all L-valued relations on a set X, a binary operation **composition** can be defined in the following way: for $\bar{S}, \bar{T} : X^2 \to L$, $\bar{S} \circ \bar{T}$ is again a mapping from X^2 to L, defined with

$$\bar{S} \circ \bar{T}(x,y) = \bigvee_{x \in X} (\bar{S}(x,z) \wedge \bar{T}(z,y)).$$

3. Partially ordered fuzzy sets. Completion

If $(P, \leq)$ is a partially ordered set and X is a nonempty set, then any mapping $\bar{A} : X \to P$ is a **partially ordered fuzzy set** (**P-fuzzy set, P-valued**

set) on X . A p-**cut** of A, for $p \in P$, is a mapping $\bar{A}_p : X \to \{0,1\}$, so that for $x \in X$, $\bar{A}_p(x) = 1$ iff $\bar{A}(x) \geq p$. Obviously, $\bar{A}_p$ is a characteristic function of the following subset of X, called also a p-**cut**: $A_p = \{x \in X \mid \bar{A}(x) \geq p\}$.

Every P-valued fuzzy set on X induces a partition of the poset P, as shown in the sequel (the proofs can be found in [2]).

Let $\bar{A} : X \to P$ be a P-fuzzy set on X, and $\approx$ a binary relation on P, such that for $p, q \in P$ $p \approx q$ iff $A_p = A_q$. $\approx$ is an equivalence relation on P. Let $\bar{A}(X) := \{p \in P \mid p = \bar{A}(x)$, for some $x \in X\}$, and for $p \in P$, let $[p) = \{q \in P \mid p \leq q\}$. We shall now state the main properties of the above partition and of the ordering induced on it.

Lemma 1. *If $\bar{A}$ is a P-fuzzy set on X and $p, q \in P$, then*

a) $p \approx q$ iff $[p) \cap \bar{A}(X) = [q) \cap \bar{A}(X)$;

b) $[p]_\approx \leq [q]_\approx$ iff $A_q \subseteq A_p$;

c) if $p = \bar{A}(x)$, then $p = \bigvee [p]_\approx$. □

Now we shall formulate the properties of the colletion of levels for a P-fuzzy set, including theorems for the decomposition and synthesis.

Proposition 3. *If $\bar{A}$ is a P-fuzzy set on X, then for $x \in X$*

$$\bar{A}(x) = \bigvee (p \in P \mid \bar{A}_p(x) = 1)$$

(which means that the join on the right exists in $(P, \leq)$ for every $x \in X$ and is equal to $\bar{A}(x)$). □

Let $A_P := \{A_p \mid p \in P\}$, for $\bar{A}$ on X.

Proposition 4. *If $\bar{A}$ is a P-fuzzy set on X, then*

a) if $p, q \in P$ and $p \leq q$, then for every $x \in X$ $\bar{A}_q(x) \leq \bar{A}_p(x)$;

b) if for $Q \subseteq P$ there exists a supremum of Q $(\bigvee (p \mid p \in Q))$, then

$$\bigcap (A_p \mid p \in Q) = A_{\bigvee (p \mid p \in Q)};$$

c) $\bigcup (A_p \mid p \in P) = X$;

d) for every $x \in X$, $\bigcap (A_p \mid x \in A_p) \in A_P$. □

Proposition 5. *Let X be a nonempty set and P a family of its subsets with the following properties:*

(i) $\bigcup P = X$;

(ii) for every $x \in X$, $\bigcap(p \in P \mid x \in p) \in P$.

Let $\bar{A} : X \to P$ *be defined with*

$$\bar{A}(x) := \bigcap(p \in P \mid x \in p).$$

Then, $\bar{A}$ *is a* P*-fuzzy set, where* $(P, \leq)$ *is a partially ordered set under* $p \leq q$ *iff* $q \subseteq p$ $(p, q \in P)$, *and for every* $p \in P$, $p = A_p$. □

Proposition 4 gives basic properties of the poset of levels A_P of a P-fuzzy set $\bar{A}$. On the other hand, Proposition 5 claims that precisely these properties are necessary in order that a collection of subsets (characteristic functions) of a set X represents the poset A_P of a P-fuzzy set $\bar{A}$ on X. We can start with an arbitrary collection $\mathcal{F}$ of subsets of X and look for a P-fuzzy set $\bar{A}$ on X, such that $\mathcal{F}$ can be embedded into the poset of levels of $\bar{A}$. There is always such fuzzy set obtained simply by taking the power set of X to be the collection of levels. In the following, we prove that, if the union of $\mathcal{F}$ is X, there is a fuzzy set whose poset of levels is minimal among those that contain $\mathcal{F}$ as a subset of the poset of levels.

Lemma 2. *Let $\mathcal{F}$ be a collection of subsets of a nonempty set X. If* $a, b \in \mathcal{F}$ *and* $a \in \bigcap\{f \in \mathcal{F} \mid b \in f\}$, *then*

$$\bigcap\{f \mid a \in f\} \subseteq \bigcap\{f \mid b \in f\}. \tag{2}$$

Proof. Let $x \in \bigcap\{f \mid a \in f\}$. This statement is equivalent with the following: $(\forall f \in \mathcal{F})(a \in f \to x \in f)$. Similarly, by the assumption, $(\forall f \in \mathcal{F})(b \in f \to a \in f)$. Hence, if $b \in f$ then $x \in f$, which proves that $x \in \bigcap\{f \mid b \in f\}$. Thus, (2) holds. □

Theorem 1. *Let $\mathcal{F}$ be the collection of subsets of a nonempty set X, union of which is X. Then there is a P-fuzzy set $\bar{A}$ on X with the minimal poset of levels containing $\mathcal{F}$ (i.e. if $\mathcal{F} \subseteq B_P$ for a P-fuzzy set $\bar{B}$ on X, then $A_P \subseteq B_P$).*

Proof. Let P be the collection of subsets of X obtained by adding to $\mathcal{F}$, for every $x \in X$, the intersection of all members of $\mathcal{F}$ to which x belongs. Since the union of $\mathcal{F}$ is X, every x is in some member of $\mathcal{F}$, and this intersection can be uniquely determined. Obviously, $\mathcal{F} \subseteq P$. P is thus a collection of subsets of X which satisfies conditions of Proposition 5, hence there is a P-fuzzy set $\bar{A}$ having P as a poset of levels, i.e. for which $A_P = P$. Thus, A_P contains precisely the elements of $\mathcal{F}$ and, for every $x \in X$, the intersection of all members of $\mathcal{F}$ to which it belongs. By Proposition 4, every P-fuzzy set $\bar{B}$ on X, for which $\mathcal{F} \subseteq B_P$ has among its levels the members of A_P, proving that $\bar{A}$ has a minimal collection of levels. □

Theorem 1 gives an algorithm for the construction of a fuzzy set $\bar{A}$ on X, with a particular collection $\mathcal{F}$ as a subset of A_P, provided that the union of $\mathcal{F}$ is X. In addition, $\bar{A}$ has minimal such collection of levels. We shall say that $\bar{A}$ is a **minimal P-fuzzy completion** (or simply a **P-completion** of the collection $\mathcal{F}$).

If $\mathcal{F}$ is an arbitrary (nonempty) collection of subsets of X (union of which is only a subset of X), then there is no minimal collection of level subsets containing $\mathcal{F}$, and it is not possible to apply the above algorithm. However, since there are P-fuzzy sets containing $\mathcal{F}$ as a subset of levels, we shall define a **P-completion** of $\mathcal{F}$ in the following way.

Let X be a nonempty set and $\mathcal{F}$ a collection of its subsets. Let

$$\mathcal{F}' = \mathcal{F} \cup X.$$

$\mathcal{F}$' satisfies conditions of Theorem 1. The minimal P-fuzzy completion of $\mathcal{F}$' will be called a **P-completion** of the collection $\mathcal{F}$.

Thus, for every collection $\mathcal{F}$ of subsets of X, there is a unique P-fuzzy set $\bar{A}$ on X, which is a P-completion of $\mathcal{F}$.

4. Partially ordered relations

A **partially ordered fuzzy relation** (**P-fuzzy relation, P-valued relation**) on a nonempty set X is a P-fuzzy set on X^2. Hence, we do not have to define p-cuts, neither to formulate the properties analoguos to the ones listed above, including decomposition and synthesis.

A P-fuzzy relation on X is **reflexive** (**symmetric, transitive**) if all its p-cuts are ordinary reflexive (symmetric, transitive) relations.

A P-fuzzy relation on X is **antisymmetric** if all its p-cut are antisymmetric relations, except X^2 , if the latter belongs to the collection of p-cuts. A P-**similarity relation** (P-**equivalence**) on X is a reflexive, symmetric and transitive P-valued relation on X. A P-**quasi-ordering** on X is reflexive and transitive, and P-**ordering** is a P-quasi ordering, which is also an antisymmetric P-valued relation on X.

We need the following definition, already stated for L-valued relations. If a relation (P-fuzzy or ordinary) or every relation in a collection satisfies some of the above properties which are not explicitely listed, we shall say that this relation or all the relations in a collection have some **special properties**.

Some particular properties of P-fuzzy relations are given in the following proposition, the proof of which can be found in [6].

Proposition 6. *Let $\bar{R}$ be a P-fuzzy relation on X.*

a) If $\bar{R}$ is reflexive, then there is the greatest element 1 in P, and $\bar{R}(x,x) = 1$, for all $x \in X$;

b) If $\bar{R}$ is symmetric, then $\bar{R}(x,y) = \bar{R}(y,x)$, for all $x, y \in X$;

c) If $\bar{R}$ is transitive, then $\bar{R}(x,y) \wedge \bar{R}(y,z) \leq \bar{R}(x,z)$, provided that $\bar{R}(x,y) \wedge \bar{R}(y,z)$ exists;

d) If R is antisymmetric, and for some $x, y \in X$, $x \neq y$, the meet $\bar{R}(x,y) \wedge \bar{R}(y,x)$ exists, then there is the least element 0 in P and $\bar{R}(x,y) \wedge \bar{R}(y,x) \in 0$ (i.e. the meet is the element of the least class of $P/\approx$). □

The synthesis of a special P-fuzzy relation by a collection of ordinary relations, all having the same special properties, is a consequence of a general property of P-fuzzy sets. Note that the full relation on X, namely X^2, can be included in any collection of relations having special properties (even if one of them is antisymmetry), because X^2 can appear in a collection of levels of any P-fuzzy relation.

Proposition 7. *Let $\mathcal{F} = \{\rho_i \mid i \in I\}$ be a collection of (ordinary) relations on a set X, all having some special properties. Necessary and sufficient conditions under which $\mathcal{F}$ represent a family of level relations of a P-fuzzy relations satisfying the same special properties are:*

(i) for every pair $(x,y) \in X^2$, $\bigcap\{\rho_i \in \mathcal{F} \mid (x,y) \in \rho_i\} \in \mathcal{F}$;

(ii) for every pair $(x,y) \in X^2$, there is $\rho_i \in \mathcal{F}$ such that $x\rho y$.

Proof. Let $\mathcal{F}$ be a family of relations on X, all having some special properties, and let (i) and (ii) hold. By Proposition 5 there is a P-fuzzy set $\bar{R}$ on X^2, i.e. a P-fuzzy relation on X, for which $\mathcal{F}$ is a collection of p-cuts. By the definition, $\bar{R}$ has the same special properties as all the relations in $\mathcal{F}$.

On the other hand, if $\bar{R}$ is a P-fuzzy relation on X with some special properties, then it's family of p-cut contains relations with the same properties, and it satisfies (i) and (ii), by Proposition 4. □

In the practical application of relations, it is sometimes necessary to deal with a collection of (ordinary) relations satisfying special properties (collection of equivalences, quasi-orderings etc.). If we want to express them by a single P-fuzzy relation, they must satisfy conditions of Proposition 5. Thus, an arbitrary collection of that kind has to be completed in the sense of Theorem 1 and the definition P-completion.

Theorem 2. *If $\mathcal{F} = \{\rho_i \mid i \in I\}$ is a family of (ordinary) relations on X, all having some special properties, then their P-completion is a P-fuzzy relation satisfying the same special properties.*

Proof. By the definition, a P-completion is obtained by adding to the given family X^2 (if it is not already there), and for every pair $(x, y) \in X^2$, the intersection of all relations from $\mathcal{F}$ containing (x, y). Now, each particular special property (namely: reflexivity, symmetry, antisymmetry and transitivity) is preserved under intersections, and X^2 satisfies all of them (including antisymmetry, by the remark before Proposition 7). Hence, the added relations also satisfy the same special property, and all the relations in the new collection satisfy conditions for the synthesis (Proposition 7). Thus, the P-fuzzy relation obtained by the synthesis satisfies the special properties, which was to be proved. □

In the classical theory of relations the following construction is well known and often appears in applications. If R is a quasi-ordering relation on the set X, then the relation S, defined with

$$xSy \text{ iff } (xRy \text{ and } yRx) \tag{3}$$

is an equivalence relation on X. In addition, let T be the relation defined on the set of equivalence classes of S as follows:

$$[x]T[y] \text{ iff } xRy.$$

Then, T is an ordering relation on X.

A similar construction (quasi-order, equivalence, order), has been done in [8] for L-valued relations. For P-valued relations this construction fails, since the obtained level equivalences do not satisfy conditions for the synthesis. To overcome this problem, we shall use a P-completion of collections of (ordinary) relations. For the sake of simplicity in formulations, we shall suppose that for P-fuzzy sets $\bar{A} : X \to P$ (and also for P-valued relations), the classes in $P/\approx$ (defined before Lemma 1) are one-element sets. In that case, the poset P is order-antiisomorphic with the collection of p-cuts of $\bar{A}$ under set inclusion (this anti-isomorphism is the mapping $p \mapsto A_p$).

Theorem 3. *Let $\bar{R}$ be a P-quasi-ordering relation on X. For every $p \in P$, let S_p be an ordinary relation on X, defined as follows:*

$$(4) \qquad x S_p y \ \textit{ iff } \ x R_p y \ \textit{ and } \ y R_p x.$$

Then, the P-completion $\bar{S}$ of the family $\{S_p \mid p \in P\}$ is a P-fuzzy equivalence on X.

Proof. By (3), for every $p \in P$, S_p is an equivalence on X. By the preceding theorem, relations constructed by the P-completion of the family $\{S_p \mid p \in P\}$ are again equivalence relations on X. All these equivalences together satisfy conditions for the synthesis, and the obtained relation $\bar{S}$ is a P-fuzzy equivalence on X. □

The family $\{S_p \mid p \in P\}$ from the above theorem contains some levels of the partially ordered equivalence $\bar{S}$, precisely the ones introduced by (4). The poset P is only an index-set for that family, levels are not determined by these indices. Connection of P with the codomain of $\bar{S}$ is given by the following proposition.

Proposition 8. *Under conditions of Theorem 3, for $p, q \in P$,*

$$p \leq q \textit{ implies } S_q \subseteq S_p$$

(i.e. the mapping $p \mapsto S_p$ is an order reversing homomorphism of the poset P (the codomain of $\bar{R}$) into the poset of levels of $\bar{S}$.

Proof. If $p, q \in P$ and $p \leq q$, then $R_q \subseteq R_p$, since P is the domain of $\bar{R}$, and R_p, R_q are its levels. By (4), $S_q \subseteq S_p$. □

As for L-fuzzy relations, we can define a partition, corresponding to a P-equivalence. If $\bar{S}$ is a P-valued equivalence relation on X and $a \in X$, then the P-fuzzy set $[a]_{\bar{S}}$ on X, defined with

$$[a]_{\bar{S}}(x) := \bar{S}(a, x),\ x \in X, \tag{5}$$

is a **(P-valued) $\bar{S}$-equivalence class** of a. The collection $\prod_{\bar{S}}$ of all $\bar{S}$-equivalence classes, is a **P-valued partition** of X, induced by $\bar{S}$:

$$\prod\nolimits_{\bar{S}} := \{[x]_{\bar{S}} \mid x \in X\}.$$

Lemma 3. *If $\prod_{\bar{S}}$ is a P-valued partition induced by a P-valued equivalence on X, then for every $p \in P$, collection of levels $\{([x]_{\bar{S}})_p \mid x \in X\}$ is an ordinary partition of X.*

Proof. Indeed, for every $p \in P$, the level relation $\bar{S}_p$ is an ordinary equivalence on X, and for every $x \in X$ the block of the partition induced by $\bar{S}_p$ is the set $([x]_{\bar{S}})_p$: by the definition of a p-cut and by (4), $y \in ([x]_{\bar{S}})_p$ iff $[x]_{\bar{S}}(y) \geq p$ iff $\bar{S}(x, y) \geq p$ iff xS_py. □

Now we can prove that a P-valued quasi-order induces an order on every level partition of the corresponding P-valued equivalence.

Theorem 4. *Let P be a poset, $\bar{R}$ a P-valued quasi-order on X and $\bar{S}$ the corresponding partially ordered equivalence, constructed by the completion (Theorem 3). Let also $\prod_{\bar{S}}$ be the partially ordered partition induced by $\bar{S}$. Then, for every $p \in P$, $\bar{R}$ induces an order on the partition of levels from $\prod_{\bar{S}}$, indexed by p.*

Proof. By Theorem 3 and Proposition 8, the family $\{S_p \mid p \in P\}$ consists of levels of $\bar{S}$, indexed by elements of P. By (4), for every $p \in P$ we can define a relation T_p on the partition - collection of levels in $\prod_{\bar{S}}$:

$$([x]_{\bar{S}})_p T_p ([y]_{\bar{S}})_p \text{ iff } xR_py.$$

Since R_p is a quasi-order and S_p an ordinary equivalence on X, the proof that T_p is an order on the corresponding partition follows by the well known properties of (ordinary) relations. □

Now we shall define a composition of P-valued relations on the same set. Let $\bar{R}$ and $\bar{S}$ be two P-valued relations on X. A P-**composition** of $\bar{R}$ and $\bar{S}$ is a P-valued relation $\bar{R} \circ \bar{S}$ on X, defined in the following way. Let

$$\{R_p \circ S_p \mid p \in P\} \tag{6}$$

be the family of (ordinary) compositions of (ordinary) p-levels of $\bar{R}$ and $\bar{S}$. We define $\bar{R} \circ \bar{S}$ to be a P-completion of that family. (It is necessary to complete the family (6), since (in general) it violates conditions for the synthesis of P-valued relations.

In the following, we show that the P-composition is closely related to the transitivity, just as in the case of ordinary relations.

Proposition 9. *Let P be a poset and $\bar{R} \circ \bar{S}$ a P-composition of P-valued relations $\bar{R}, \bar{S}$ on X. Then, the mapping $p \mapsto R_p \circ S_p$ is an order reversing homomorphism from P into the poset of levels of $\bar{R} \circ \bar{S}$.*

. *Proof.* Let $p, q \in P$, $p \leq q$. Then $R_q \subseteq R_p$ and $S_q \subseteq S_p$, by the definition of levels. Hence, $R_q \circ S_q \subseteq R_p \circ S_p$, since the ordinary composition preserves the set inclusion. □

Now let P be a poset and $\bar{R}$ a P-valued relation on X. By the above proposition, the function $p \mapsto R_p \circ R_p$ is an order reversing homomorphism from P into the collection of levels of $\bar{R} \circ \bar{R}$. This collection of levels is, by Proposition 5 and Theorem 1, the codomain P_1 of $\bar{R} \circ \bar{R}$. If the elements from P are identified with their homomorphic images in P_1, then $\bar{R}$ can be taken to be a partially ordered relation codomain of which is also P_1. Under such assumptions, we can prove the following theorem.

Theorem 5. *Let P be a poset and $\bar{R}$ a P-valued relation on the set X. Let also $\bar{R} \circ \bar{R}$ be a partially ordered composition, as a mapping from X^2 to the poset P_1. Then, $\bar{R}$ is P-transitive if and only if $\bar{R} \circ \bar{R} \subseteq \bar{R}$, where the codomain of $\bar{R}$ is taken to be P_1.*

Proof. Let $\bar{R}$ be P-transitive, and for $(x, y) \in X^2$, let $\bar{R} \circ \bar{R}(x, y) = p$. By the definition of the P-completion, there are three possibilities: (i) p is a homomorphic image of an element from P; (ii) p is obtained by the P-completion of the collection of levels $\{R_q \circ R_q \mid q \in P\}$; (iii) p corresponds X^2.

(*i*) If p is a homomorphic image of some element from P, then, since all levels of $\bar{R}$ are transitive relations, $R_p \circ R_p \subseteq R_p$, i.e. $\bar{R} \circ \bar{R}(x,y) \leq \bar{R}(x,y)$.

(*ii*) If p is obtained by the P-completion, then it is the intersection of some levels in $\{R_q \circ R_q \mid q \in P\}$. For all these levels $R_q \circ R_q \subseteq R_q$, and again $\bar{R} \circ \bar{R} \subseteq \bar{R}$.

(*iii*) If p is the element of P_1 corresponding to X^2, then $\bar{R} \circ \bar{R} \subseteq \bar{R}$ since p is then the least element in P_1.

Now let $\bar{R} \circ \bar{R} \subseteq \bar{R}$ in P_1. Then all the levels of $\bar{R}$ in P_1 are transitive relations, proving that $\bar{R}$ is transitive as the mapping from X^2 to P_1. □

5. Relational valued special relations

Relational valued fuzzy sets are introduced in [3], and relational valued relations in [6]. They represent the most general approach to fuzziness via level subsets, since any collection of subsets of a set can be used for the synthesis of a fuzzy set of that kind. Here we use the above concept of a completion to enable a transition from relational valued to partially ordered special relations. We advance necessary definitions and propositions, all taken from [3] and [6].

Let $\mathcal{S} = (S, \rho)$ be a relational system where S is a nonempty set and ρ is a binary relation on S, such that for all $a, b \in S$

(7) $$a \neq b \text{ iff } \{x \in S \mid (x,a) \in \rho\} \neq \{x \in S \mid (x,b) \in \rho\}.$$

ρ is said to have the **unique projection property** (ρ is a **UP-relation**), if it satisfies the above condition. In addition, $\mathcal{S}$ is a **UP-relational system.** Let now X be a nonempty set and $\mathcal{S} = (S, \rho)$ a UP-relational system. A **relational valued fuzzy set** (**R-valued set, R-fuzzy set**) on X is a mapping $\bar{A} : X \to S$. For $p \in S$, a **p-cut** of $\bar{A}$ is a subset A_p of X, defined with

(8) $$A_p := \{x \in X \mid (p, \bar{A}(x)) \in \rho\}.$$

The characteristic function of A_p is denoted by $\bar{A}_p$ (also a **p-cut** of $\bar{A}$).

R-valued fuzzy sets can be uniquely represented by their families of p-cuts (that is why the UP-relational systems are introduced). Moreover, any collection $\mathcal{F}$ of characteristic functions on a finite set X can be used for the synthesis of an R-valued fuzzy set, as shown by the algorithm that follows (for more details and all proofs, see [3]):

Let $X = \{0,1\}^n$, and let $\mathcal{F} = \{f_1, ..., f_n\} \subseteq X$. Let H be the matrix whose rows are the functions from $\mathcal{F}$. Let m be the number of different columns in H. Let $A = \{1, 2, ..., n\}$ and let S be an arbitrary set with r elements, where $r = max\{k, m\}$, say, $S = \{p_1, ..., p_r\}$. For $i = 1, ..., r$, let $\bar{A}_{p_i} : A \to \{0,1\}$, and $\bar{A}_{p_i} = f_i$, if $k \geq m$; if $m > k$, then for every $i = 1, ..., k$, again $\bar{A}_{p_i} = f_i$, and for $i = k+1, ..., m$, $\bar{A}_{pi} = f_k$.

We shall construct an R-fuzzy set $\bar{A} : X \to S$, so that $\{\bar{A}_{p_i} \mid p_i \in S\}$ becomes its family of p-cuts:

$\bar{A}(1) = p_1$, and recursively, for $j \in \{2, ..., n\}$: if the j-th column in H is equal to the i-th one for some $i < j$, then $\bar{A}(j) \doteq \bar{A}(i)$; otherwise $\bar{A}(j) = p_{s+1}$ where s is the greatest index such that p_s is the value of some $\bar{A}(i)$, $i < j$.

Finally, we have to define $\rho \subseteq S$. For $p_i, p_j \in S$,

$$(p_i, p_j) \in \rho \text{ iff } \bar{A}_{p_i}(k) = 1, \text{ where } \bar{A}(k) = p_j.$$

Since equal columns in H correspond to the same element of S, ρ is a UP-relation on S.

Thus, for any collection $\mathcal{F}$ of subsets of a set X, there is a relational valued fuzzy set, levels of which are precisely the sets from $\mathcal{F}$. Obviously, $\mathcal{F}$ is a poset under the set inclusion. Starting with this poset and using the P-completion, we can construct a P-fuzzy set.

Proposition 10. *Let $\bar{A}$ be an R-valued fuzzy set on X, and $\mathcal{F}$ the collection of its levels. Then there is a partially ordered fuzzy set on X, whose poset of levels contains all the elements of $\mathcal{F}$.*

Proof. Let us construct a P-completion $\mathcal{F}_1$ of $\mathcal{F}$. Thus we obtain a P fuzzy set, again on X, with the collection of levels $\mathcal{F}_1$. This is the required P-valued set, since, by the construction, $\mathcal{F} \subseteq \mathcal{F}_1$. □

***R*-valued relations** on X are R-valued sets on X^2 (functions $X^2 :\to S$, where (S, ρ) is UP-relational system).

We shall introduce special properties exactly in the same way as for P-valued relations, i.e. using levels.

An R-fuzzy relation on X is **reflexive (symmetric, transitive)** if all its p-cuts are ordinary reflexive (symmetric, transitive) relations.

An R-fuzzy relation on X is **antisymmetric** if all its p-cut are antisymmetric relations, except X^2 , if the latter belongs to the collection of p-cuts. An R-**equivalence** relation on X is reflexive, symmetric and transitive, an R-**quasi-ordering** is reflexive and transitive and R-**ordering** is reflexive, antisymmetric and transitive relation on X.

An R-valued relation on X, satisfying some of the above properties is said to have **special properties**.

Theorem 6. *For any finite collection $\mathcal{F}$ of (ordinary) relations on X which satisfy some special properties, there is a UP-relational system (S, ρ) and an R-fuzzy relation R on X satisfying the same special properties, such that $\mathcal{F}$ represents its family of level functions (p-cuts).*

Proof. If we apply the above algorithm on the given collection of relations, all satisfying the same special properties, we shall obtain a relational valued fuzzy set on X^2. By the definition, this fuzzy set is a relational valued relation having special properties, which was to be proved. □

Finally, we shall prove that the P-completion enable a construction of a partially ordered special relation, if a relational valued one is given.

Theorem 7. *For any R-valued special relation $\bar{R}$ on X, there is a partially ordered relation on the same set X, which satisfies the same special properties, and whose collection of levels contans all the levels of $\bar{R}$.*

Proof. Let $\mathcal{F}$ be the collection of levels of $\bar{R}$, and $\mathcal{F}_1$ the corresponding collection of subsets of X^2, obtained by the P-completion of $\mathcal{F}$. By Theorem 2 and Proposition 10, $\mathcal{F}_1$ is a collection of levels of a partially ordered relation on X, having precisely the same special properties as R, and among whose levels are all the relations from $\mathcal{F}$. □

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REZIME

PARCIJALNO UREDJENE I RELACIONO VREDNOSNE RASPLINUTE RELACIJE II

U radu se posmatraju rasplinute relacije kao preslikavanja iz kvadrata proizvoljnog nepraznog skupa u parcijalno uredjeni skup (P-rasplinute relacije) i u relacioni sistem (R-rasplinute relacije). Nastavljaju se istraživanja iz rada [6], definisanjem specijalnih relacija preko nivo-relacija. Kako se neki važni pojmovi teorije relacija ne mogu uvesti samo posmatranjem nivoa, u ovom radu uvodi se postupak P-kompletiranja kolekcije podskupova datog skupa, tako da oni zadovoljavaju uslove za sintezu rasplinutog (parcijalno uredjenog) skupa. Tako je moguće ispitivati kompoziciju rasplinutih relacija i ekvivalenciju odnosno poredak, koji se izvode iz rasplinutog pretporetka.

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