Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 24, 1 (1994), 23-30

Review of Research
Faculty of Science
Mathematics Series

FOUR MAPPINGS WITH A COMMON FIXED POINT

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Abstract

A common fixed point theorem satisfying a symmetric rational expresion has been proved which, in turn, unifies some fixed point theorems of Fisher and Khan. An example for illustration is also included.

AMS Mathematics Subject Classification (1991): 54H25, 47H10
Key words and phrases: common fixed point, coincidence point, weakly commuting pair.

1. Introduction

Fisher [1] has extended the Banach contraction principle through a symmetric rational expresion and obtained the following result which in turn modifies the theorem of Khan [3].

Theorem 1. *Let (X,d) be a complete metric space and T a self-mapping on X such that for all x, y in X either*

$$d(Tx,Ty) \leq k\{\frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{d(x,Ty)+d(y,Tx)}\}$$

if $d(x, Ty) + d(y, Tx) \neq 0$, *where* $0 \leq k < 1$ *or*

$$d(Tx, Ty) = 0$$

if $d(x, Ty) + d(y, Tx) = 0$.
Then T *has a unique fixed point.*

Quiet recently Khan-Swaleh-Imdad [4] has unified BanachContraction Principle and Theorem 1. The purpose of this paper is to unify the theorem of Fisher [2] and Theorem 1. Our unification is two fold: Firstly it extends Theorem 1 to a common fixed point theorem for four mappings; secondly, it generalizes the theorem of Fisher [2].

While proving our theorem, we employ a notion of weak commutativity due to Sessa [5] which runs as follows:

Definition 1. *A pair of self-mappings* $\{S, I\}$ *of a metric space* (X, d) *is said to be weakly commuting if* $d(SIx, ISx) \leq d(Ix, Sx)$ *for all* x *in* X.

It is obvious that two commuting mappings are weakly commuting but the opposite is not true as shown in Example 1 of Sessa [5].

2. Result

We prove the following:

Theorem 2. *Let* $\{S, I\}$ *and* $\{T, J\}$ *be weakly commuting pair of mappings of a complete metric space* (X, d) *into itself such that*

(1) $T(X) \subset I(X)$, $S(X) \subset J(X)$. *And for all* x, y *in* X;

Either

(2) $d(Sx, Ty) \leq \alpha\{\frac{d(Ix,Sx)d(Ix,Ty)+d(Jy,Ty)d(Jy,Sx)}{d(Ix,Ty)+d(Jy,Sx)}\} + \beta d(Ix, Jy)$
if $d(Ix, Ty) + d(Jy, Sx) \neq 0$, *where* α, $\beta > 0$, $\alpha + \beta < 1$, *or*

(2') $d(Sx, Ty) = 0$ *if* $d(Ix, Ty) + d(Jy, Sx) = 0$.

If one of S, T, I *or* J *is continuous then* S, T, I *and* J *have an unique common fixed point* z. *Further* z *is the unique common fixed point of* S *and* I *as well as of* T *and* J.

Proof. Let x_0 be an arbitrary point of X. Since $S(X) \subset J(X)$ we can find a point x_1 in X such that $Sx_0 = Jx_1$. Also, since $T(X) \subset I(X)$ we can further choose a point x_2 with $Tx_1 = Ix_2$. In general for the point x_{2n} we can pick up a point x_{2n+1} such that $Sx_{2n} = Jx_{2n+1}$ and then a point x_{2n+2} with $Tx_{2n+1} = Ix_{2n+2}$ for $n = 0, 1, 2, ...$

Let us put $U_{2n} = d(Sx_{2n}, Tx_{2n+1})$ and $U_{2n+1} = d(Tx_{2n+1}, Sx_{2n+2})$. Now we distinguish the two cases:

(i) Suppose $U_{2n} \neq 0$, $U_{2n+1} \neq 0$ for $n = 0, 1, 2, ...$
Then on using inequality (2), we have

(3) $U_{2n+1} \leq (\alpha + \beta)^{2n+1} U_0$, for $n = 0, 1, 2, ...$
It follows that the sequence

(4) $\{Sx_0, Tx_1, Sx_2, ..., Tx_{2n-1}, Sx_{2n}, Tx_{2n+1}, ...\}$
is a Cauchy sequence in the complete metric space (X, d) and so gets a limit point z in X. Hence the sequences $\{Sx_{2n}\} = \{Jx_{2n+1}\}$ and $\{Tx_{2n-1}\} = \{Ix_{2n}\}$ which are subsequences of (4) also converge to the same point z.

Let us now suppose that I is continuous so that the sequences $\{I^2 x_{2n}\}$ and $\{ISx_{2n}\}$ converge to the same point Iz. Since S and I are weakly commuting, we have

$$d(SIx_{2n}, ISx_{2n}) \leq d(Ix_{2n}, Sx_{2n})$$

and so the sequence $\{SIx_{2n}\}$ also converges to the point Iz.

We now have

$$\begin{aligned} d(SIx_{2n}, Tx_{2n+1}) \quad \leq \quad & \alpha\{\frac{d(I^2x_{2n}, SIx_{2n})d(I^2x_{2n}, Tx_{2n+1})}{d(I^2x_{2n}, Tx_{2n+1}) + d(Jx_{2n+1}, SIx_{2n})} \\ & + \frac{d(Jx_{2n+1}, Tx_{2n+1})d(Jx_{2n+1}, SIx_{2n})}{d(I^2x_{2n}, Tx_{2n+1}) + d(Jx_{2n+1}, SIx_{2n})}\} \\ & + \beta d(I^2x_{2n}, Jx_{2n+1}) \end{aligned}$$

which on letting $n \to \infty$ reduces to

$$d(Iz, z) \leq \beta d(Iz, z),$$

giving thereby $Iz = z$.
Further,

$$\begin{aligned} d(Sz, Tx_{2n+1}) \quad \leq \quad & \alpha\{\frac{d(Iz, Sz)d(Iz, Tx_{2n+1}) + d(Jx_{2n+1}, Tx_{2n+1})d(Jx_{2n+1}, Sz)}{d(Iz, Tx_{2n+1}) + d(Jx_{2n+1}, Sz)}\} \\ & + \beta d(Iz, Jx_{2n+1}), \end{aligned}$$

which on making n tend to infinity gives $d(Sz, z) = 0$ and hence $Sz = z$.

Since $Sz = z$ and $S(X) \subset J(X)$ there always exists a point z' such that $Jz' = z$. Thus

$$\begin{aligned} d(z, Tz') &= d(Sz, Tz') \\ &\leq \alpha\{\frac{d(Iz, Sz)d(Iz, Tz') + d(Jz', Tz')d(Jz', Sz)}{d(Iz, Tz') + d(Jz', Sz)}\} + \beta d(Iz, Jz') \\ &= 0, \end{aligned}$$

giving thereby $Tz' = z$.

Since T and J weakly commute

$$d(Tz, Jz) = d(TJz', JTz') \leq d(Jz', Tz') = d(z, z) = 0,$$

which yields $Tz = Jz$ and so

$$\begin{aligned} d(z, Tz) &= d(Sz, Tz) \\ &\leq \alpha\{\frac{d(Iz, Sz)d(Iz, Tz) + d(Jz, Tz)d(Jz, Sz)}{d(Iz, Tz) + d(Jz, Sz)}\} + \beta d(Iz, Jz) \\ &= \beta d(z, Tz), \end{aligned}$$

which implies that $z = Tz = Jz$.

Thus we have proved that z iz a common fixed point of S, T, I and J.

Now suppose that S is continuous, so that the sequences $\{S^2x_{2n}\}$ $\{SIx_{2n}\}$ converge to the point Sz. Since S and I weakly commute, it follows as earlier that the sequence $\{ISx_{2n}\}$ also converges to the Sz. Thus

$$\begin{aligned} d(S^2x_{2n}, Tx_{2n+1}) \leq\ & \alpha\{\frac{d(ISx_{2n}, S^2x_{2n})d(ISx_{2n}, Tx_{2n+1})}{d(ISx_{2n}, Tx_{2n+1}) + d(Jx_{2n+1}, S^2x_{2n})} \\ & + \frac{d(Jx_{2n+1}, Tx_{2n+1})d(Jx_{2n+1}, S^2x_{2n})}{d(ISx_{2n}, Tx_{2n+1}) + d(Jx_{2n+1}, S^2x_{2n})}\} \\ & + \beta d(ISx_{2n}, Jx_{2n+1}), \end{aligned}$$

which on letting $n \to \infty$ gives

$$d(Sz, z) \leq \beta d(Sz, z),$$

implying thereby $Sz = z$.

As $S(X) \subset J(X)$ and $Sz = z$, once again we can find a point z' in X such that $Jz' = z$. Thus

$$d(S^2x_{2n}, Tz') \leq \alpha\{\frac{d(ISx_{2n}, S^2x_{2n})d(ISx_{2n}, Tz') + d(Jz', Tz')d(Jz', S^2x_{2n})}{d(ISx_{2n}, Tz') + d(Jz', S^2x_{2n})}\} + \beta d(ISx_{2n}, Jz').$$

Making $n \to \infty$, we get $d(z, Tz') = 0$ so that $Tz' = z$.

Since T and J are weakly commuting, it again follows as above that $Tz = Jz$. Further

$$d(Sx_{2n}, Tz) \leq \alpha\{\frac{d(Ix_{2n}, Sx_{2n})d(Ix_{2n}, Tz) + d(Jz, Tz)d(Jz, Sx_{2n})}{d(Ix_{2n}, Tz) + d(Jz, Sx_{2n})}\} + \beta d(Ix_{2n}, Jz),$$

which on making $n \to \infty$, gives $Tz = z$.

Thus the point z is in the range of T and since the range of I contains the range of T, there always exists a point z'' in X such that $Iz'' = z$. Thus

$$\begin{aligned} d(Sz'', z) &= d(Sz'', Tz) \\ &\leq \alpha\{\frac{d(Iz'', Sz'')d(Iz'', Tz) + d(Jz, Tz)d(Jz, Sz'')}{d(Iz'', Tz) + d(Jz, Sz'')}\} + \beta d(Iz'', Jz) \\ &= 0, \end{aligned}$$

yielding thereby $Sz'' = z$.

Again since S and I weakly commute, we have

$$d(Sz, Iz) = d(SIz'', ISz'') \leq d(Iz'', Sz'') = d(z, z) = 0.$$

Thus $Sz = Iz = z$.

We have thus proved again that z is a common fixed point of S, T, I and J.

If the mapping T or J is continuous instead of S or I, then the proof that z is a common fixed point of S, T, I and J is similar.

To show that z is unique, let w be a second common fixed point of S and I, then

$$\begin{aligned} d(w, z) &= d(Sw, Tz) \\ &\leq \alpha\{\frac{d(Iw, Sw)d(Iw, Tz) + d(Jz, Tz)d(Jz, Sw)}{d(Iw, Tz) + d(Jz, Sw)}\} + \beta d(Iw, Jz) \\ &\leq \beta d(w, z), \end{aligned}$$

giving thereby $w = z$.

Similarly, it can be proved that z is a unique common fixed point of T and J.

(ii) If $U_{2n} = 0$ for some n, then the inequality (3) gives $U_{2n+1} = 0$ which implies that

$$Sx_{2n} = Jx_{2n+1} = Tx_{2n+1} = Ix_{2n+2} = Sx_{2n+2} = \dots = z.$$

Now we assert that there exists a point w such that $Sw = Iw = Tw = z$, otherwise if $Sw = Tw \neq z$, then

$$\begin{aligned} 0 < d(Iw, z) &= d(Sw, Tx_{2n+1}) \\ &\leq \alpha\{\frac{d(Iw, Sw)d(Iw, Tx_{2n+1})}{d(Iw, Tx_{2n+1}) + d(Jx_{2n+1}, Sw)} \\ &\quad + \frac{d(Jx_{2n+1}, Tx_{2n+1})d(Jx_{2n+1}, Sw)}{d(Iw, Tx_{2n+1}) + d(Jx_{2n+1}, Sw)}\} + \beta d(Iw, Jx_{2n+1}) \\ &= \beta d(Iz, z), \end{aligned}$$

which yields that $Iw = Sw = z$. Similarly, one can argue that $Tw = Jw = z$.

Now, suppose I or S is continuous, then proceeding in the similar way, it can be shown that $Iw = z$ is a unique common fixed point of S, T, I and J. Similarly if J or T is continuous, the proof that z is a unique common fixed point of S, T, I and J is similar. This completes the proof.

Remark 1. If we choose $\beta = 0$ and $S = I = J = T$, then Theorem 2 reduces to the theorem of Fisher [1] which, in turn, corrects the theorem of Khan [3].

Remark 2. If we set $\alpha = 0$ then Theorem 2 gives a modified form of the theorem of Fisher [2] for two pairs of weakly commuting mappings. Note that the theorem of Fisher [2] involves only a triod of mappings.

Remark 3. By choosing α, β, I, J, S and T suitably, we can derive a multitude of fixed point theorems which already exist in the literature. We omit the details.

Remark 4. Theorem 2 ensures that S, I, T and J have a unique common fixed point. However, either S or I or T or J may have other fixed point. One can note that in our Example 1 S and J have two and three fixed points respectively.

Remark 5. It follows from the proof of Theorem 2 that if condition (2') is omitted from the statement of Theorem 2 then we can say that z is a coincidence of S, I, T and J.

3. An example

Finally, we adapt the following example for the illustration of Theorem 2, which also indicates the degree of generality of our extension.

Example 3. Let $X = \{A, B, C, D\}$ be a finite set of R^2 with Euclidean metric d, where $A \equiv (0,0)$, $B \equiv (0,2)$, $C \equiv (1,0)$ and $D \equiv (0,1/4)$. Then clearly (X, d) is a complete metric space.

Now define S, I, T and J on X as follows:

$$SA = SB = SD = A, \quad SC = C$$

$$IA = IB = A, \quad IC = B, \quad ID = C$$

$$TA = TB = TC = A, \quad TD = C$$

$$JA = A, \quad JB = JD = B, \quad JC = C$$

Note that $S(X) = \{A, C\} \subset \{A, B, C\} = J(X)$ and $T(X) = \{A, C\} \subset \{A, B, C\} = I(X)$.
Since

$$SIA = A = ISA, \quad SIB = A = IS, \quad 2 = d(SIC, ISC) \leq d(IC, SC) = \sqrt{5},$$

$1 = d(SID, ISD) \leq d(ID, SD) = 1$ whereus
$JTA = A = TJA$, $JTB = A = TJB$, $JTC = A = TJC$,
$2 = d(TJD, JTD) \leq d(JD, ID) = \sqrt{5}$, the pairs $\{S, I\}$ and $\{T, J\}$ are weakly commuting.

Further, a routine calculation shows that inequality (2) holds with, for instance, $\alpha = \beta = 40/100$. Therefore all the conditions of Theorem 2 are satisfied and A is the unique common fixed point of S, I, T and J. Also it can be noted that A is the unique common fixed point of S, I and that of T and J.

However, Theorem 2 is a genuine extension of the theorem of Fisher [2] because if we choose $x = B \equiv (0,2)$, $y = C \equiv (1,0)$ then the condition

$d(Sx,Sy) \leq kd(Ix,Jy)$ implies that $1 \leq k$ which is a contradiction to the fact that $0 \leq k < 1$.

Acknowledgement. The authors are grateful to Mr. A. Rauf Khan for his helpful comments regarding the presentation of the paper.

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REZIME

ČETIRI PRESLIKAVANJA SA ZAJEDNIČKOM NEPOKRETNOM TAČKOM

Dokazana je teorema o zajedničkoj nepokretnoj tački, u obliku simetričnog racionalnog izraza, koja objedinjuje neke Fisherove i Khanove teoreme o nepokretnoj tački. Takodje je dat i ilustrativni primer.

Received by the editors October 25, 1990