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THE SPACES OF WEIGHTED AND TEMPERED ULTRADISTRIBUTIONS
Part II

Dušanka Kovačević
Institute of Mathematics, University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

In the paper, the structure of spaces $\mathcal{D}'^{(M_\alpha)}_{L^r,\mu}(R)$ $(r \in [1,\infty],\ \mu \in R)$ of weighted ultradistributions of the Beurling type and the space $\mathcal{S}'^{(M_\alpha)}(R)$ of tempered ultradistribution of the Beurling type is investigated, as well as the dual spaces of the spaces of weighted ultradifferentiable functions and the space of rapidly decreasing ultradifferentiable functions, which are defined in [4], their relation with the known spaces of distributions and ultradistributions and the properties of the elementary operations in those spaces.

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1. Introduction

Following the approach of Komatsu ([3]) to the theory of Beurling ultradistributions, we define and investigate the spaces $\mathcal{D}'^{(M_\alpha)}_{L^r,\mu}(R)$ $(r \in [1,\infty],\ \mu \in R)$ of weighted ultradistributions of the Beurling type on R, the space $S'^{(M_\alpha)}(R)$ of tempered ultradistributions of the Beurling type on R and the space $O_M{}^{(M_\alpha)}(R)$ as the natural generalizations of the spaces $\mathcal{D}'_{L^r}(R)$,

$\mathcal{S}'(R)$, $\mathcal{O}_M(R)$ ([10]), $\mathcal{D}'^{(M\alpha)}_{L^r}(R)$ ([7] and [8]), $\mathcal{D}'_{L^s,\mu}(R)$ ([5]) and $S'_{(M\alpha)}$ ([9]).

In the first section are given the structure theorems for the spaces of weighted ultradistributions of the Beurling type and the space of tempered ultradistributions of the Beurling type; the embedding relations between these spaces and the known spaces of distributions and ultradistributions are described. It is proved that the spaces $\mathcal{D}'^{(M\alpha)}_{L^r,\mu}(R)$, $r \in [1,\infty)$ are complete, semireflexive, ultrabornological spaces, that they are reflexive if $r \in (1,\infty)$ and that the space $\mathcal{S}'^{(M\alpha)}(R)$ is a complete, bornological, Montel space.

The properties of differentiation, ultradifferentiation and multiplication by a smooth function in the spaces of weighted and tempered ultradistributions are investigated in the second section. It is proved that the spaces $\mathcal{D}'^{(M\alpha)}_{L^r,\mu}(R)$ and $S'^{(M\alpha)}(R)$ are stable under differential resp. ultradifferential operators, if condition (M.2)' resp. (M.2) (see below) is fulfilled. The basic properties of the pointwise multiplication in these spaces are obtained and it is proved that the space $O^{(M\alpha)}_M(R)$, defined in [4], is the space of multipliers of the space $\mathcal{S}'^{(M\alpha)}(R)$.

Let us give a survey of definitions needed in the paper.

By N we denote the set of non-negative integers.

Throughout the paper we will suppose that $(M\alpha)_{\alpha\in N}$ is a sequence of positive numbers, which satisfy the following conditions:

(M.1) $$M_\alpha^2 \le M_{\alpha-1}M_{\alpha+1}, \ \ \alpha = 1,2,\dots$$

(M.3)' $$\sum_{\alpha=1}^{\infty} \frac{M_{\alpha-1}}{M_\alpha} < \infty, \ \ \alpha = 1,2,\dots$$

In some assertions we will suppose the next condition

(M.2)' There are constants A and H such that

$$M_{\alpha+1} \le AH^\alpha M_\alpha, \ \ \alpha = 1.2,\dots$$

which will sometimes be replaced by the following stronger condition

(M.2) There are constants A and H such that

$$M_\alpha \le AH^\alpha \min_{0\le\beta\le\alpha} M_\beta M_{\alpha-\beta}, \ \ \alpha \in N.$$

$$\mathcal{E}^{M\alpha,h}([-\rho,\rho]) = \{\varphi \in C^\infty(R), \sum_{\alpha\in N} \frac{h^\alpha \|\varphi^{(\alpha)}\|_{\infty},[-\rho,\rho]}{M_\alpha} < \infty\},$$

$h > 0, \ \rho > 0,$

$$\mathcal{E}^{(M\alpha)}(R) = \underset{\rho\to\infty}{\text{proj lim}} \ \underset{h\to\infty}{\text{proj lim}} \ \mathcal{E}^{M\alpha,h}([-\rho,\rho]),$$

$$\mathcal{D}^{M\alpha,h}_{[-\rho,\rho]} = \{\varphi \in C^\infty(R), \text{supp}\varphi \subset [-\rho,\rho], \ \sum_{\alpha\in N} \frac{h^\alpha \|\varphi^{(\alpha)}\|_{\infty}, \ [-\rho,\rho]}{M_\alpha} < \infty\},$$

$h, \rho > 0,$

$$\mathcal{D}^{(M\alpha)}_{[-\rho,\rho]} = \underset{h\to\infty}{\text{proj lim}} \ \mathcal{D}^{M\alpha,h}_{[-\rho,\rho]},$$

$$\mathcal{D}^{(M\alpha)}(R) = \underset{\rho\to\infty}{\text{ind lim}} \ \mathcal{D}^{(M\alpha)}_{[-\rho,\rho]}.$$

By $\mathcal{D}'^{(M\alpha)}(R)$ resp. $\mathcal{E}'^{(M\alpha)}(R)$ is denoted the strong dual of the space $\mathcal{D}^{(M\alpha)}(R)$ resp. $\mathcal{E}^{(M\alpha)}(R)$.

A differential operator $P(D) = \sum_{\alpha\in N} a_\alpha D^\alpha$, where $D := \frac{1}{i}\frac{d}{dx}$, $i := (-1)^{1/2}$ and $a_\alpha \in C$, is caled an ultradifferential operator of the class $(M\alpha)$, whenever the coefficients satisfy the estimate $|a_\alpha| \leq cL^\alpha/M_\alpha$, $\alpha \in N$, for some costants c and L.

The usual norm in $L^s(R)$, $s \in [1,\infty]$, is denoted by $\| \ \|_s$.

The spaces $\mathcal{D}^{(M\alpha)}_{L^s}(R)$, $s \in [1,\infty]$, and $\dot{\mathcal{B}}^{(M\alpha)}(R)$ were defined in [7] and [8].

$$\mathcal{D}^{M\alpha,h}_{L^s}(R) = \{\varphi \in C^\infty(R), \gamma_{s,h}(\varphi) = \sum_{\alpha\in N} \frac{h^\alpha \|\varphi^{(\alpha)}\|_s}{M_\alpha} < \infty\}, \ h > 0,$$

$$\mathcal{D}^{(M\alpha)}_{L^s}(R) = \underset{h\to\infty}{\text{proj lim}} \ \mathcal{D}^{M\alpha,h}_{L^s}(R),$$

$\dot{\mathcal{B}}^{(M\alpha)}(R)$ is a subspace of $\mathcal{B}^{(M\alpha)}(R) := \mathcal{D}^{(M\alpha)}_{L^\infty}(R)$, which is the completion of $\mathcal{D}^{(M\alpha)}(R)$ under the family of norms $\gamma_{\infty,h}$, $h > 0$.

By $\mathcal{D}'^{(M\alpha)}_{L^r}(R)$, $r \in (1,\infty]$, resp. $\mathcal{D}'^{(M\alpha)}_{L^1}(R)$ we denote the strong dual

$$\mathcal{D}^{(M\alpha)}_{L^s}(R) = \underset{h\to\infty}{\text{proj lim}} \ \mathcal{D}^{M\alpha,h}_{L^s}(R),$$

$\dot{\mathcal{B}}^{(M\alpha)}(R)$ is a subspace of $\mathcal{B}^{(M\alpha)}(R) := \mathcal{D}^{(M\alpha)}_{L^\infty}(R)$, which is the completion of $\mathcal{D}^{(M\alpha)}(R)$ under the family of norms $\gamma_{\infty,h}$, $h > 0$.

By $\mathcal{D}'^{(M\alpha)}_{L^r}(R)$, $r \in (1,\infty]$, resp. $\mathcal{D}'^{(M\alpha)}_{L^1}(R)$ we denote the strong dual space of the space $\mathcal{D}^{(M\alpha)}_{L^s}(R)$, $(1/s) + (1/r) = 1$, resp. $\dot{\mathcal{B}}^{(M\alpha)}(R)$.

Let $\mu \in R$. Recall [5], the space

$$\mathcal{D}'_{L^s,\mu}(R) = \{f \in \mathcal{D}'(R),\ < x > ^{\mu} f \in \mathcal{D}'_{L^s}(R)\},\ < x > := (1 + |x|^2)^{1/2},$$

is equipped with the topology, which is induced by the bijection

$$\mathcal{D}'_{L^s}(R) \longrightarrow \mathcal{D}'_{L^s,\mu}(R), \quad f \longmapsto < x > ^{-\mu} f.$$

Let $\mu \in R$ and $h > 0$.

$\mathcal{D}^{M\alpha,h}_{L^s,\mu}(R)$, $s \in [1,\infty]$, (see [4]) is the space of all the functions $\varphi \in C^\infty(R)$, such that $(< x > ^{\mu}\varphi)^{(\alpha)} \in L^s(R)$, for each $\alpha \in N$, and that

$$\| \varphi \|_{\mathcal{D}^{M\alpha,h}_{L^s,\mu}(R)} := \gamma_{s,h}(< x > ^{\mu}\varphi) = \sum_{\alpha \in N} \frac{h^\alpha \|(< x > ^{\mu}\varphi)^{(\alpha)}\|_s}{M_\alpha} < \infty,$$

equipped with the topology induced by the norm $\| \ \|_{\mathcal{D}^{M\alpha,h}_{L^s,\mu}(R)}$.

$$\mathcal{D}^{(M\alpha)}_{L^s,\mu}(R) := \lim_{h \to \infty} \operatorname{proj} \mathcal{D}^{M\alpha,h}_{L^s,\mu}(R).$$

$$\mathcal{B}^{(M\alpha)}_{\mu}(R) := \mathcal{D}^{(M\alpha)}_{L^\infty,\mu}(R).$$

$\dot{\mathcal{B}}^{(M\alpha)}_{\mu}(R)$ is the subspace of $\mathcal{B}^{(M\alpha)}_{\mu}(R)$ which is the completion of $\mathcal{D}^{(M\alpha)}$ (R) under the family of norms $\| \ \|_{\mathcal{D}^{M\alpha,h}_{L^\infty,\mu}(R)}$, $h > 0$.

$$\mathcal{S}^{(M\alpha)}(R) = \operatorname{proj} \lim_{\mu \to \infty} \mathcal{D}^{(M\alpha)}_{L^s,\mu}(R).$$

The notation "$A \hookrightarrow B$", means that the space A is dense in the space B and that the inclusion mapping $i : A \to B$ is continuous.

2. Structural properties

Definition 1. *Let $r \in (1, \infty]$ and $(1/s) + (1/r) = 1$. We denote by $\mathcal{D}'^{(M\alpha)}_{L^r,\mu}(R)$, $\mathcal{D}'^{(M\alpha)}_{L^1,\mu}(R)$, $S'^{(M\alpha)}(R)$ the strong duals of the space $\mathcal{D}'^{(M\alpha)}_{L^s,-\mu}(R)$, $\dot{\mathcal{B}}^{(M\alpha)}_{-\mu}(R)$, $S^{(M\alpha)}(R)$ respectively.*

In order to prove the structure theorems for the spaces $\mathcal{D}'^{(M\alpha)}_{L^r,\mu}(R)$, $r \in [1, \infty]$, and $S'^{(M\alpha)}(R)$, we need the next two assertions.

Lemma 1. *If $(E_\alpha, \| \ \|_\alpha)$, $\alpha \in \{0, 1, \dots \beta\}$, are normed spaces and the topology on $E = \prod_{\alpha=0}^{\beta} E_\alpha$ is induced by the norm*

$$|||x||| = \sum_{\alpha \le \beta} \|x_\alpha\|_\alpha, \quad x = (x_0, x_1, \dots, x_\beta) \in E,$$

then f is a continuous linear functional on E if and only if for each $\alpha \le \beta$ there exists $f_\alpha \in E'_\alpha$ such that

$$< f, x >= \sum_{\alpha \le \beta} < f_\alpha, x_\alpha > . \tag{1}$$

The norm of f is determined by

$$|||f|||' = \sum_{\alpha \le \beta} \|f_\alpha\|'_\alpha. \tag{2}$$

Proof. From [1 p.60, 4.Satz] it follows (1). Let us determine the norm of f. Since for each $x \in E$

$$| < f, x > | = | \sum_{\alpha \le \beta} < f_\alpha, x_\alpha > | \le \sum_{\alpha \le \beta} | < f_\alpha, x_\alpha > | \le \sum_{\alpha \le \beta} \|f_\alpha\|'_\alpha \, \|x_\alpha\|_\alpha,$$

we have

$$|||f|||' \le \sum_{\alpha \le \beta} \|f_\alpha\|'_\alpha. \tag{3}$$

For a given $\varepsilon > 0$ there exists $x = (x_0, x_1, \dots, x_\beta) \in E$ such that $|||x||| \le 1$ and that

$$\|f_\alpha\|'_\alpha - \varepsilon/\beta \le | < f_\alpha, x_\alpha > |, \quad \alpha \in \{0, 1, \dots, \beta\}$$

If $x^o_\alpha := x_\alpha \operatorname{sgn} < f_\alpha, x_\alpha >$, then $(x^o_0, x^o_1, \ldots, x^o_\beta) \in E$, $\|x^o\| \leq 1$ and

$$\sum_{\alpha\leq\beta} \|f_\alpha\|_\alpha - \varepsilon \leq \sum_{\alpha\leq\beta} | < f_\alpha, x_\alpha > | = \sum_{\alpha\leq\beta} | < f_\alpha, x^o_\alpha > | = < f, x^o > \leq$$

$$\leq \|f\| \ \|x^o\| \leq \|f\|.$$

From above it follows that for each $\varepsilon > 0$

$$\sum_{\alpha\leq\beta} \|f_\alpha\|'_\alpha - \varepsilon \leq \|f\|',$$

which together with (3) implies (2). □

Lemma 2. *Let $s \in [1, \infty)$, $(1/r) + (1/s) = 1$ and $T_s(h)$ be the space of all the sequences $G = (G_\alpha)_{\alpha\in N}$ of elements from $L^s(R)$ with the norm*

$$\|G\| = \sum_{\alpha\in N} \frac{h^\alpha}{M_\alpha} \|G_\alpha\|_s < \infty. \tag{4}$$

The continuous linear functional F on $T_s(h)$ has the form

$$F(G) = \sum_{\alpha\in N} \int_R F_\alpha(x) G_\alpha(x)\, dx, \tag{5}$$

where $F_\alpha \in L^r(R)$, $\alpha \in N$, and the norm of F is determined by

$$\|F\|' = \begin{cases} \sum_{\alpha\in N} \frac{M_\alpha}{h^\alpha} \|F_\alpha\|'_r, & s \in (1, \infty) \\ \\ \sup_{\substack{\alpha \in N \\ x \in R}} \frac{M_\alpha}{h^\alpha} |F_\alpha|, & s = 1 \end{cases}. \tag{6}$$

Proof. Let F be a continuous linear functional on $T_s(h)$, $G = (G_0, G_1, \ldots, G_\alpha, \ldots) = (G_\alpha)_{\alpha\in N} \in T_s(h)$ and g_α, $\alpha \in N$, be the sequence $(0, 0, \ldots, 0, G_\alpha, 0, \ldots)$. In the space $T_s(h)$, it holds that

$$G = \sum_{\alpha\in N} g_\alpha.$$

Indeed, for each $\varepsilon > 0$ and large enough $\beta = \beta(\varepsilon) \in N$

$$\|G - \sum_{\alpha\leq\beta} g_\alpha\| = \|\sum_{\alpha>\beta} g_\alpha\| = \sum_{\alpha>\beta} \frac{h^\alpha}{M_\alpha} \|G_\alpha\|_s < \varepsilon.$$

From the continuity of F it follows that

$$F(G) = F(\sum_{\alpha \in N} g_\alpha) = \sum_{\alpha \in N} F(g_\alpha) = \sum_{\alpha \in N} F((0, \ldots, 0, G_\alpha, 0, \ldots, 0)).$$

For each $\gamma \in N$ there exists F_γ a continuous linear functional on $L^s(R)$, such that for each $(G_\alpha)_{\alpha \in N} \in T_s(h)$

$$F((0, \ldots, 0, G_\alpha, 0, \ldots, 0)) = F_\gamma(G_\gamma).$$

Therefore for each $G = (G_\alpha)_{\alpha \in N} \in T_s(h)$

$$F(G) = \sum_{\alpha \in N} F_\alpha(G_\alpha), \tag{7}$$

where F_α, $\alpha \in N$, is a continuous linear functional on $L^s(R)$. We derive (5) from (7), using Riesz's representation of a continuous linear functional on $L^s(R)$

Let us prove (6). We have

$$|F(G)| = |\sum_{\alpha \in N} \int_R F_\alpha(x) G_\alpha(x) dx| \leq \sum_{\alpha \in N} \|F_\alpha(x) G_\alpha(x)\|_1 \leq$$

$$\leq \sum_{\alpha \in N} \frac{M_\alpha}{h^\alpha} \|F_\alpha(x)\|_r \cdot \sum_{\alpha \in N} \frac{h^\alpha}{M_\alpha} \|G_\alpha(x)\|_s,$$

which follows, in the case $s \in (1, \infty)$, from the Hölder inequality and, in the case $s = 1$, from [2, p. 176, (6.8) Theorem]. This implies

$$\|F\|' \leq \sum_{\alpha \in N} \frac{M_\alpha}{h^\alpha} \|F_\alpha(x)\|_r. \tag{8}$$

Let $\beta \in N$ be fixed and let the topology in the space $\prod_{\alpha \leq \beta} L^s(R)$ be induced by the norm

$$|||(H_\alpha)_{\alpha \leq \beta}||| = \sum_{\alpha \leq \beta} \frac{M_\alpha}{h^\alpha} \|H_\alpha(x)\|_s, \quad (H_\alpha)_{\alpha \leq \beta} \in \prod_{\alpha \leq \beta} L^s(R).$$

According to Lemma 1 by

$$< f, (H_\alpha)_{\alpha \leq \beta} >= \sum_{\alpha \leq \beta} < F_\alpha, H_\alpha >, \quad (H_\alpha)_{\alpha \leq \beta} \in \prod_{\alpha \leq \beta} L^s(R),$$

is defined a continuous linear functional on $\prod_{\alpha\leq\beta} L^s(R)$ and the norm of f is determined by

$$|||f|||' = \sum_{\alpha\in N} \frac{M_\alpha}{h^\alpha}\|F_\alpha\|'_r.$$

Therefore, for each $\varepsilon > 0$ there exists $(H^o_\alpha)_{\alpha\leq\beta}$ such that

$$\|(H^o_\alpha)_{\alpha\leq\beta}\| \leq 1$$

and that

$$\sum_{\alpha\leq\beta} \frac{M_\alpha}{h^\alpha}\|F_\alpha\|'_r - \varepsilon = |||f|||' - \varepsilon \leq |< f, (H^o_\alpha)_{\alpha\leq\beta} >| = |\sum_{\alpha\leq\beta} < F_\alpha, H^o_\alpha >|.$$

Since $(H^o_0, H^o_1, \dots H^o_\beta, 0, 0, \dots) \in T_s(h)$, it follows from (7) that

$$F((H^o_0, H^o_1, \dots H^o_\beta, 0, 0, \dots)) = \sum_{\alpha\leq\beta} F_\alpha(H^o_\alpha) + \sum_{\alpha>\beta} F_\alpha(0) = \sum_{\alpha\leq\beta} F_\alpha(H^o_\alpha)$$

it follows

$$|\sum_{\alpha\leq\beta} < F_\alpha, H^o_\alpha >| = |F((H^o_0, H^o_1, \dots H^o_\beta, 0, 0, \dots))| \leq$$

$$\|F\|' \cdot \|(H^o_0, H^o_1, \dots H^o_\beta, 0, 0, \dots)\| = \|F\|' \sum_{\alpha\leq\beta} \frac{h^\alpha}{M_\alpha}\|H^o_\alpha\|_s =$$

$$\|F\|' \quad |||(H^o_\alpha)_{\alpha\leq\beta}||| \leq \|F\|'.$$

The above holds for each $\beta \in N$ and $\varepsilon > 0$, hence

$$\sum_{\alpha\in N} \frac{M_\alpha}{h^\alpha}\|F_\alpha(x)\|_r \leq \|F\|',$$

which together with (8) implies (6) . □

Theorem 1. *Let $r \in [1, \infty]$ and $(1/s) + (1/r) = 1$.*

(i) An element f from $\mathcal{D}'^{(M\alpha)}(R)$ belongs to $\mathcal{D}'^{(M\alpha)}_{L^r,\mu}(R)$ if there exist $h > 0$ and a sequence $(F_\alpha)_{\alpha\in N}$ of elements from $L^r(R)$ such that

(9) $$< f, \varphi >= \sum_{\alpha\in N} < (< x >^{-\mu} F_\alpha)^{(\alpha)}, \varphi >, \quad \varphi \in \mathcal{D}^{(M\alpha)}_{L^s,-\mu}(R)$$

and

(10) $$\sum_{\alpha\in N}\frac{M_\alpha}{h^\alpha}\|F_\alpha(x)\|_r<\infty$$

(ii) Conversely, if for a sequence $(F_\alpha)_{\alpha\in N}$ of elements of $L^r(R)$ (10) holds, then f defined by (9) belongs to $\mathcal{D}'^{(M\alpha)}_{L^r,\mu}(R)$.

Proof. Let $r\in(1,\infty]$, $(1/r)+(1/s)=1$, and let $\overline{\mathcal{D}}^{M\alpha,h}_{L^s,-\mu}(R)$ be the closure of $\mathcal{D}^{(M\alpha)}_{L^s,-\mu}(R)$ in the space $\mathcal{D}^{M\alpha,h}_{L^s,-\mu}(R)$. Note, $(\mathcal{D}^{(M\alpha)}_{L^s,-\mu}(R))'=\bigcup_h(\overline{\mathcal{D}}^{M\alpha,h}_{L^s,-\mu}(R))'$ (see proof of [1,p.47, 2.2.Satz]). If $f\in\mathcal{D}'^{(M\alpha)}_{L^r,\mu}(R)$, then it follows, by the Hahn-Banach theorem, that there exists $h>0$ such that f has a continuous linear extension $\overline{f}$ on $\overline{\mathcal{D}}^{M\alpha,h}_{L^s,-\mu}$. with the same dual norm. Let $T_s(h)$ be the space defined as in Lemma 2. The mapping

$$j:\overline{\mathcal{D}}^{M\alpha,h}_{L^s,-\mu}\longrightarrow T_s(h),\qquad \varphi\longmapsto((-1)^\alpha<x>^{-\mu}\varphi^{(\alpha)})_\alpha$$

is an isometry of $\overline{\mathcal{D}}^{M\alpha,h}_{L^s,-\mu}$ onto $P_s(h)=j(\overline{\mathcal{D}}^{M\alpha,h}_{L^s,-\mu})\subset T_s(h)$. We define a continuous linear functional F on $P_s(h)$ by

$$<F,\,(G_\alpha)_\alpha>=<\overline{f},\,j^{-1}((G_\alpha)_\alpha)>,\qquad (G_\alpha)_\alpha\in P_s(h).$$

Again by the Hanh-Banach theorem we extend F on $T_s(h)$ linearly and continuously with the same dual norm and we denote this extension by $\overline{F}$. Lemma 2 implies that there exists a sequence $(F_\alpha)_\alpha$ of elements from $L^r(R)$ such that $\overline{F}$ has a form

$$<\overline{F},\,(G_\alpha)_\alpha>=\sum_{\alpha\in N}\int_R F_\alpha(x)G_\alpha(x)dx,\quad (G_\alpha)_\alpha\in T_s(h)$$

and the norm of $\overline{F}$ is given by

$$\|\overline{F}\|'=\sum_{\alpha\in N}\frac{M_\alpha}{h^\alpha}\|F_\alpha(x)\|_r.$$

Thus

$$\sum_{\alpha\in N}\frac{M_\alpha}{h^\alpha}\|F_\alpha(x)\|_r=\|\overline{F}\|'=\|f\|'<\infty$$

and if $\varphi\in\mathcal{D}^{(M\alpha)}_{L^s,-\mu}(R)$ we have

$$<f,\varphi>=<\overline{f},\varphi>=<F,\,((-1)^\alpha<x>^{-\mu}\varphi^{(\alpha)})_\alpha>=$$

$$< \overline{F}, ((-1)^\alpha < x > ^{-\mu}\varphi^{(\alpha)})_\alpha >= \sum_{\alpha\in N}\int_R F_\alpha(x)(-1)^\alpha < x > ^{-\mu}\varphi^{(\alpha)}dx =$$

$$= \sum_{\alpha\in N} < (< x > ^{-\mu}F_\alpha)^{(\alpha)}, \varphi >,$$

which implies the first part of the theorem. The second part of the proof is obvious.

The case $r = 1$ requires only a slight modification of the previous proof. □

Theorem 2. *Let $r \in (1, \infty]$.*

(i) An element f from $\mathcal{D}'^{(M\alpha)}(R)$ belongs to $S'^{(M\alpha)}(R)$ if there exist $h > 0$ and a sequence $(F_\alpha)_{\alpha\in N}$ of elements from $L^r(R)$ such that in $S'^{(M\alpha)}$

$$f = \sum_{\alpha\in N}(< x > ^{-\mu}F_\alpha)^{(\alpha)} \tag{11}$$

and

$$\sum_{\alpha\in N}\frac{M_\alpha}{h^\alpha}\|F_\alpha(x)\|_r < \infty \tag{12}$$

(ii) Conversely, if for a sequence $(F_\alpha)_{\alpha\in N}$ of elements of $L^r(R)$, (12) holds then f defined by (11) belongs to $S'^{(M\alpha)}(R)$.

Proof. We have $S'^{(M\alpha)}(R) = \bigcup_{h\in N}(S_h^{M\alpha,h}(R))'$ (see [4 Corollary 2.9.]). Applying the same idea as in the proof of the previous theorem one can prove this assertion. □

[4, Proposition 2.4.] implies the next assertion.

Theorem 3. *If $p \in [1, \infty]$, the mapping*

$$\mathcal{D}'^{(M\alpha)}_{L^p,\mu}(R) \longrightarrow \mathcal{D}'^{(M\alpha)}_{L^p}(R), \quad f \mapsto < x > ^{\mu}f$$

is an homomorphism.

Theorem 4. *Let $p, q \in [1, \infty]$, $\mu, \nu \in R$.*

(i) If $(p \leq q, \nu \leq \mu)$ or $(q < p,\ \nu < \mu - ((1/q) - (1/p))$, then

$$\begin{array}{ccccccccc}
\mathcal{D}'(R) & \hookrightarrow & S'(R) & \hookrightarrow & \mathcal{D}'_{L^p,\mu}(R) & \hookrightarrow & \mathcal{D}'_{L^q,\nu}(R) & \hookrightarrow & \mathcal{E}'(R) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{D}'^{(M\alpha)}(R) & \hookrightarrow & S'^{(M\alpha)}(R) & \hookrightarrow & \mathcal{D}'^{(M\alpha)}_{L^p,\mu}(R) & \hookrightarrow & \mathcal{D}'^{(M\alpha)}_{L^q,\nu}(R) & \hookrightarrow & \mathcal{E}'^{(M\alpha)}(R).
\end{array}$$

(ii) $\mathcal{D}'_{L^p,\mu}(R)$ is a proper subset of $\mathcal{D}'^{(M\alpha)}_{L^p,\mu}(R) \bigcap \mathcal{D}'(R)$.

Proof. As an immediate consequence of [4, Theorem 2.10.] and [4, Corollary 3.4.] we have (i).

In [8, p.291] it is proved that

$$\sum_{\alpha\in N} \frac{\delta^{(\alpha)}(\cdot - \alpha)}{M_\alpha} \in (\mathcal{D}'^{(M\alpha)}_{L^p}(R) \bigcap \mathcal{D}'(R)) \backslash \mathcal{D}'_{L^p}(R).$$

Therefore, from Theorem 3.6 and the definition of $\mathcal{D}'_{L^p,\mu}(R)$ and $\mathcal{D}'(R)$ it follows that

$$\sum_{\alpha\in N} \frac{\delta^{(\alpha)}(\cdot - \alpha)}{M_\alpha} \in (\mathcal{D}'^{(M\alpha)}_{L^p,\mu}(R) \bigcap \mathcal{D}'(R)) \backslash \mathcal{D}'_{L^p,\mu}(R). \quad \square$$

Theorem 5. *(i) If $r \in (1, \infty)$ the space $\mathcal{D}'^{(M\alpha)}_{L^r,\mu}(R)$ is reflexive .*

(ii) If $r \in [1, \infty)$ the space $\mathcal{D}'^{(M\alpha)}_{L^r,\mu}(R)$ is complete, ultrabornological and semireflexive.

(iii) The space $S'^{(M\alpha)}(R)$ is a complete, bornological, Montel space.

Proof. 1^o In order to prove assertion (i), it is enough to prove that if $s \in (1, \infty)$, the space $\mathcal{D}^{(M\alpha)}_{L^s,\mu}(R)$ is a reflexive space. The space $\mathcal{D}^{(M\alpha)}_{L^s,\mu}(R)$ is a Fréchet space (see [4]) and so it is a barrelled space. Therefore, it is enough to prove that $\mathcal{D}^{(M\alpha)}_{L^s,\mu}(R)$ is semireflexive.

The space $\prod_{\alpha\in N} L^s(R)$ is, as a product of reflexive spaces, reflexive. Since, $\mathcal{D}^{M\alpha,h}_{L^s,\mu}(R)$, $h > 0$, is isomorphic to a closed subspace of $\prod_{\alpha\in N} L^s(R)$, where the isomorphism is given by

$$j : \varphi \mapsto \left(\frac{h^\alpha}{M_\alpha} < x >^{\mu} \varphi^{(\alpha)}\right)_{\alpha\in N}, \quad \varphi \in \mathcal{D}^{M\alpha,h}_{L^s,\mu}(R),$$

it is semireflexive. It follows that $\mathcal{D}^{(M\alpha)}_{L^s,\mu}(R)$ is a projective limit of semireflexive spaces. This implies that $\mathcal{D}^{(M\alpha)}_{L^s,\mu}(R)$ is semireflexive.

2^o Analogously as in the proof of [5, Proposition 9] one can prove (ii).

3^o Since $\mathcal{S}^{(M\alpha)}(R)$ is a strict $(\mathrm{F}\overline{\mathrm{S}})$–space (see [4]) its dual is a complete bornological, Montel space (see [1]). □

3. Operations in the Space of Weighted and Tempered Ultradistributions

3.1. (Ultra)differentiation

Theorem 6. *Let $P^*(D) = \sum_{\alpha\in N} a_\alpha(-1)^\alpha D^\alpha$ be an ultradifferential operator of the class $(M\alpha)$.*

(i) Suppose that (M.2)' is fulfilled $r \in [1,\infty]$ and $(1/s)+(1/r)=1$. The operators

(13) $$D^\beta : \mathcal{D}'^{(M\alpha)}_{L^r,\mu}(R) \longrightarrow \mathcal{D}'^{(M\alpha)}_{L^r,\mu}(R),\ \beta\in N,$$

(14) $$D^\beta : \mathcal{S}'^{(M\alpha)}(R) \longrightarrow \mathcal{S}'^{(M\alpha)}(R),\ \beta\in N,$$

(15) $$P(D): \mathcal{D}'_{L^r,\mu}(R) \longrightarrow \mathcal{D}'^{(M\alpha)}_{L^r,\mu}(R),\ \varphi\mapsto D^\beta\varphi,\ \beta\in N,$$

(16) $$P(D): \mathcal{S}'(R) \longrightarrow \mathcal{S}'^{(M\alpha)}(R),\ \varphi\mapsto D^\beta\varphi,\ \beta\in N,$$

defined respectively as the adjoint of

(17) $$(-1)^\beta D^\beta : \mathcal{D}^{(M\alpha)}_{L^s,\mu}(R) \longrightarrow \mathcal{D}^{(M\alpha)}_{L^s,\mu}(R),\ \varphi\mapsto(-1)^\beta D^\beta\varphi,\ \beta\in N,$$

(or, if $r=1$, $(-1)^\beta D^\beta : \dot{\mathcal{B}}^{(M\alpha)}_\mu(R)\to\dot{\mathcal{B}}^{(M\alpha)}_\mu(R)$, $\varphi\mapsto(-1)^\beta D^\beta\varphi$, $\beta\in N$)

(18) $$(-1)^\beta D^\beta : \mathcal{S}^{(M\alpha)}(R) \longrightarrow \mathcal{S}^{(M\alpha)}(R),\ \varphi\mapsto(-1)^\beta D^\beta\varphi,\ \beta\in N,$$

(19) $$P^*(D): \mathcal{D}^{(M\alpha)}_{L^s,\mu}(R) \longrightarrow \mathcal{D}_{L^s,\mu}(R),$$

(or, if $r=1$ $\quad P^*(D): \dot{\mathcal{B}}^{(M\alpha)}_\mu(R)\longrightarrow\dot{\mathcal{B}}_\mu(R)$)

(20) $$P^*(D): \mathcal{S}^{(M\alpha)}(R) \longrightarrow \mathcal{S}(R),$$

respectively are continuous.

For each $f \in \mathcal{D}'_{L^r,\mu}(R)$ *resp.* $f \in \mathcal{S}'(R)$

$$P(D)f = \sum_{\alpha \in N} a_\alpha D^\alpha f, \tag{21}$$

where the series converges absolutely in $\mathcal{D}'^{(M\alpha)}_{L^r,\mu}(R)$ *resp* $S'^{(M\alpha)}(R)$ *in the sense of weak topology.*

(ii) Suppose that (M.2) is fulfilled. The operators

$$P(D):\ \mathcal{D}'^{(M\alpha)}_{L^r,\mu}(R) \longrightarrow \mathcal{D}'^{(M\alpha)}_{L^r,\mu}(R), \tag{22}$$

$$P(D):\ S'^{(M\alpha)}(R) \longrightarrow S'^{(M\alpha)}(R), \tag{23}$$

defined respectively as the duals of,

$$P^*(D):\ \mathcal{D}^{(M\alpha)}_{L^s,\mu}(R) \longrightarrow \mathcal{D}^{(M\alpha)}_{L^s,\mu}(R), \tag{24}$$

$$(\text{or, if } r = 1,\quad P^*(D):\ \dot{\mathcal{B}}^{(M\alpha)}_{\mu}(R) \longrightarrow \dot{\mathcal{B}}^{(M\alpha)}_{\mu}(R)\ )$$

$$P^*(D):\ S^{(M\alpha)}(R) \longrightarrow S^{(M\alpha)}(R). \tag{25}$$

respectively, are continuous. For each $f \in \mathcal{D}'^{(M\alpha)}_{L^r,\mu}(R)$ *resp.* $f \in S'^{(M\alpha)}(R)$ *we have (21).*

Proof. Taking into account that the image of a bounded set under a continuous linear mapping is a bounded set, one can easily see that the continuity of all the mentioned mappings follows from [4, Theorem 3.1] and [4, Theorem 3.2].

Suppose that (M.2)' resp. (M.2) is fulfilled. Let us prove (21) for $f \in \mathcal{D}'_{L^r,\mu}(R)$ resp. $f \in \mathcal{D}'^{(M\alpha)}_{L^r,\mu}(R)$. Since, for each $\varphi \in \mathcal{D}^{(M\alpha)}_{L^s,-\mu}(R)$ and $\beta \in N\backslash\{0\}$,

$$< f,\ \sum_{\alpha \le \beta} a_\alpha (-1)^\alpha D^\alpha \varphi > = < \sum_{\alpha \le \beta} a_\alpha D^\alpha f,\ \varphi >$$

converges to $< f,\ P^*(D)\varphi > = < P(D)f,\ \varphi >$ as $\beta \to \infty$, we have (21). One can analogously prove (21) for $f \in \mathcal{S}'(R)$ resp. $S'^{(M\alpha)}(R)$. □

3.2. Multiplication

Theorem 7. *Let $p, q, s \in [1, \infty]$ and $(1/p) + (1/q) \geq (1/r)$. The mappings*

$$\mathcal{D}^{(M\alpha)}_{L^p,\mu}(R) \times \mathcal{D}'^{(M\alpha)}_{L^q,\nu}(R) \longrightarrow \mathcal{D}'^{(M\alpha)}_{L^r,\mu+\nu}(R), \quad (\varphi, f) \mapsto \varphi \cdot f, \tag{26}$$

$$O^{(M\alpha)}_M(R) \times S'^{(M\alpha)}(R) \longrightarrow S'^{(M\alpha)}(R), \quad (\varphi, f) \mapsto \varphi \cdot f, \tag{27}$$

are separately continuous.

Proof. The assertion follows immediately from [4, Theorem 3.3.], [4, Theorem 3.7.] and [4, Theorem 2.10.]. □

Theorem 8. *If $\phi \in \mathcal{E}^{(M\alpha)}(R)$ is such that $\phi f \in S'^{(M\alpha)}(R)$ for all $f \in S'^{(M\alpha)}(R)$ then $\phi \in O^{(M\alpha)}_M(R)$.*

Proof. Our assumption implies that for every $\varphi \in S^{(M\alpha)}(R)$ the mapping

$$f \longmapsto < \phi f, \varphi >$$

is a continuous linear functional on $S'^{(M\alpha)}(R)$. Since $S^{(M\alpha)}(R)$ is a reflexive space (it is Montel space), there is $\psi \in S^{(M\alpha)}(R)$ such that for each $f \in S'^{(M\alpha)}(R)$

$$< \phi f, \varphi > = < f, \psi > .$$

In particular, for each $g \in \mathcal{D}^{(M\alpha)}(R)$ we have

$$< \phi g, \varphi > = < g, \psi >,$$

which implies that

$$< g, \phi\varphi > = < g, \psi > .$$

Hence $\phi\varphi = \psi \in S^{(M\alpha)}(R)$ for all $\varphi \in S^{(M\alpha)}(R)$. By [4, Remark 3.8] we get $\phi \in O^{(M\alpha)}_M(R)$. □

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REZIME

PROSTOR TEŽINSKIH I TEMPERIRANIH ULTADISTRIBUCIJA

Ispitivana je struktura prostora težinskih ultradistribucija Beurling-ovog tipa nad R, kao i prostora temperiranih distribucija Beurling-ovog tipa nad R, koji su definisani kao duali prostora težinskih ultradiferencijabilnih funkcija i prostora brzo opadajućih (u beskonačnosti) ultradiferencijabilnih funkcija (v. [4]), kao i njihovi odnosi sa poznatim prostorima distribucija i ultradistribucija i osobine elementarnih operacija u tim prostorima.

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