Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 24, 1 (1994), 159-164

Review of Research
Faculty of Science
Mathematics Series

ALGEBRAIC APPROACH TO ALMOST SURE CONVERGENCE IN L_2-SPACES

Ryszard Jajte
Institute of Mathematics, Lódź University
ul. S. Banacha 22; 90-238 Lódź, Poland

Abstract

In this paper a survey of some results of authors on the algebraic analogones of the classical theorems in the probability and ergodic theory in L_2 are given.

AMS Mathematics Subject Classification (1991): 47A35
Key words and phrases: Cesero averages, kernel, contraction, ergodic theorem

1. Introduction

In the classical probability and ergodic theory the almost sure convergence theorems for sequences in L_2 (over a probability space) belong to the most important and deep results of these theories. Let us mention here the individual ergodic theorems, the results on the almost sure convergence of orthogonal series, powers of contractions, martingales and iterates of conditional expectations. The algebraic approach to quantum statistical mechanics [1] suggests the systematic analysis of theorems just mentioned in the context of operator algebras. The classical pointwise convergence theorems for sequences in L_2 are, as a rule, non-trivial extensions of much easier results concerning the convergence in L_2-norm. The same situation is in the non-commutative case. Most of the non-commutative L_2-norm versions of the analogical classical results can be rather easily obtained by a natural modification of the classical argument. Passing to the non-commutative almost sure versions needs as a rule new methods and techniques. Very often the algebraic approach makes much clearer the general idea which is behind the results concerning, say, real functions. At the same time the proofs provide some new tools in the theory of operator algebras.

2. Basic definitions

Let (Ω, F, μ) be a probability space. $L_\infty(\Omega, F, \mu)$ can be treated as a commutative von Neumann algebra (acting in $L_2(\Omega, F, \mu)$ by multiplication).

By Egorov's theorem, the μ-almost sure convergence in $A = L_\infty(\Omega, F, \mu)$ can be expressed purely in terms of the algebra A without any reference to the base space Ω, namely, by means of the L_∞-norm, the state $\tau_\mu : f \to \int_\Omega f d\mu$ and the characteristic functions of "large" sets. This suggests the following definition.

Definition 1. *[9], [11] Let A be a von Neumann algebra with a faithful normal state Φ. We say that a sequence (x_n) of elements of A converges almost uniformly to an element $x \in A$ if, for each $\varepsilon > 0$, there is a projection $p \in A$ with $\Phi(p) > 1 - \varepsilon$ and such that $||(x_n - x)p|| \to 0$ as $n \to \infty$.*

In the sequel we consider the Hilbert space $H = L_2(A, \Phi)$ the completion of A under the norm $x \to \Phi(x * x)^{1/2}$, $x \in A$ and assume that A acts on H in a standard way, with a cyclic and separating vector ξ_0 such that $\Phi(x) = (x\xi_0, \xi_0)$, for $x \in M$.

For a projection $p \in M$ and $\xi \in H$, we put

$$S_{\xi,p} = \{(x_k) : \textstyle\sum_{k=1}^\infty x_k\xi_0 = \xi \text{ in } H \text{ and } \sum_{k=1}^\infty x_k p \text{ converges in norm in M } \}$$

and

$$||\xi||_p = \inf\{||\sum_{k=1}^\infty x_k p|| : (x_k) \in S_{\xi,p}\}.$$

We define the almost sure convergence in $H = L_2(A, \Phi)$ in the spirit of Egorov's theorem as follows.

Definition 2. *[5] A sequence (ξ_n) in H is said to be almost surely (a.s.) convergent to $\xi \in H$ if for every $\varepsilon > 0$ there exists a projection $p \in M$ such that $\phi(1-p) < \varepsilon$ and $||\xi_n - \xi||_p \to 0$ as $n \to \infty$.*

It is easily seen that in the classical commutative case of $M = L_\infty$ (over a probability space) the convergence just defined coincides with the usual almost everywhere convergence.

3. Typical results

In this section we give a survey of some results which are the algebraic analogues of the classical theorems well-known in the probability and ergodic theory in L_2. In the sequel $\alpha : A \to A$ is a linear map satisfying the conditions

$$\alpha(x * x) \geq \alpha(x) * \alpha(x), \quad \text{for all } x \in M,$$

and

$$\Phi(\alpha x) \leq \Phi(x), \quad \text{for } x \in A_+.$$

Such map α will be called a kernel.

Every kernel can be extended uniquely to a contraction β in H via the formula

$$\beta(x\xi_0) = \alpha(x)\xi_0, \quad \text{for } x \in A.$$

We say that β is generated by the kernel α.

Theorem 1. *Let β be a contraction in H generated by a kernel. Then the Cesàro averages*

$$n^{-1} \sum_{k=0}^{n-1} \beta^k \xi \to \hat{\xi} \quad a.s.,$$

for every $\xi \in H$, where $\hat{\xi}$ is given by the mean ergodic theorem for β.

Obviously, the theorem just formulated is a non-commutative version of the classical individual ergodic theorem for L_2 (for details we refer to [9], [4], [10], [7], [5].

Theorem 2. *[8] Let α_0 and β_0 be kernels in A and α and β be contractions in H generated by α_0 and β_0, respectively. Let us assume that the contraction α is positive in H (i.e. $(\alpha\xi, \xi) \geq 0$ for $\xi \in H$). Then, for every $\xi \in H$, the sequence $(\beta\alpha^n\xi)$ converges almost surely to ξ (given by the mean ergodic theorem) as $n \to \infty$.*

The above theorem (for $\beta =$ identity) is a non-commutative analogue of the classical theorem of E.M. Stein on the almost everywhere convergence of powers of positive contractions in a function space L_2.

The following theorem is a non-commutative version of the classical result of Burkholder and Chrow.

Theorem 3. *[8] Let E and F be two conditional expectations in A preserving Φ and let P and Q be orthogonal projections in H generated by E and F, respectively. Then, for every $\xi \in H$, the sequence $\{(PQ)^n\xi\}$ converges almost surely to $(P \wedge Q)\xi$ as $n \to \infty$.*

Like in the classical case, behind the individual ergodic theorems (or related results) there is always some maximal ergodic inequality. In our context the crucial point in the proofs of the above theorems is the following

Maximal ergodic lemma. *For $i = 1, 2, ..., N$, let $\alpha_{i,o}$ be a kernel in A. Denote by α_i and β_i the contractions in H generated by $\alpha_{i,o}$ and $\beta_{i,o}$, respectively. Let us put*

$$S_n^{(i)} = \frac{1}{n}\beta_i \sum_{k=0}^{n-1} \alpha_i^k, \quad \sigma_n^{(i)} = \frac{1}{n}\beta_{i,o} \sum_{k=0}^{n-1} \alpha_{i,o}^k,$$

$i = 1, 2, ..., N$. Then, for every sequence (δ_k) of positive numbers, every $(\xi_k) \subset H$ and every $(a_k) \subset A_+$, there exists a projection $p \in A$ such that

$$\Phi(1 - p) \leq 4 \sum \delta_k^{-1}(\Phi(a_k) + ||\xi_k||^2)$$

and

$$||p\sigma_n^{(i)}(a_k)p||_\infty < 2\delta_k$$

and

$$||S_n^{(i)}(\xi_k)||_p < 5\delta_k^{1/2},$$

for $n, k = 1, 2, ...$ and $i = 1, 2, ..., N$.

The proof of the last result can be reduced to the following

Goldstein's maximal ergodic theorem for several kernels.
Let $\alpha_1, \alpha_2, ..., \alpha_r, \quad \beta_1, \beta_2, ..., \beta_r$, be kernels in A, (ε_k) sequence of positive numbers, $(y_k) \subset A_+$. Set

$$S_n^{(i)} = n^{-1} \sum_{k=0}^{n-1} \alpha_i^k,$$

and

$$\sigma_n^{(i)} = \beta_i S_n^{(i)}, \quad \text{for } i = 1, 2, ..., r \ \ n = 1, 2, ...$$

Then there exists a projection $p \in A$ such that

$$\Phi(1-p) \leq 2 \sum_{n=1}^{\infty} \varepsilon_n^{-1} \Phi(y_n)$$

and

$$||p\sigma_n^{(}i)(y_k)p||_\infty \leq 2\varepsilon_k, \quad \text{for } n, k = 1, ..., r$$

(Comp. [4], [6], [7], [8]).

References

[1] Bratteli, O., Robinson, D. W., Operator algebras and quantum statistical mechanics, I. New York-Heidelberg-Berlin: Springer 1979

[2] Burkholder, D. L., Chow, Y. S., Iterates of conditional expectation operators, Proc. Am. Math. Soc. 12(1961), 490-495.

[3] Gaposhkin, V. F., Individual ergodic theorem for normal operators in l_2, (in Russian) Funkts. Anal. Appl. 15(1981), 18-22.

[4] Goldstein, M. S., Theorems in almost everywhere convergence in von Neumann algebras, (in Russian), J. Oper. Theory 6(1981), 233-311.

[5] Hensz, E., Jajte, R., Pointwise convergence theorems in L_2 over a von Neumann algebra, Math. Z. 193(1986), 413-429.

[6] Jajte, R., Strong limit theorems in non-commutative probability, (Lect. Notes Math., 1110) Berlin-Heidelberg-New York: Springer 1985

[7] Jajte, R., Ergodic theorem in von Neumann algebras, Semesterbericht Funktionenanalysis, Tübingen, Sommersemester, (1986) , 135-144.

[8] Jajte, R., Almost sure convergence of iterates of contractions in non-commutative L_2–spaces., to appear in Math. Z.

[9] Lance, E.C., Ergodic theorems for convex sets and operator algebras, Invent. Math. 37(1976), 201-214.

[10] Petz, D., Ergodic theorems in von Neumann, Acta Sci. Math. 46(1983), 329-343.

[11] Segal, I.E., A non-commutative extension of abstract integration, Ann. Math. 57 (1953)

[12] Sinai, Y.G., Aushelevich, V.V., Some problems of non-commutative ergodic theory, Russ. Math. Surv. 31(1976), 157-174.

[13] Stein, E.M., Topics in harmonic analysis, Ann. Math. Stud. no. 63. Princeton University Press 1980

REZIME

ALGEBARSKI PRILAZ SKORO SIGURNOJ KONVERGENCIJI U l_2 PROSTORU

U radu se daje pregled nekih rezultata autora o algebarskim analogonima klasičnih teorema teorije verovatnoće i ergodične teorije, u prostoru L_2.

Received by the editors October 10, 1990