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SOME NEW PROPERTIES OF THE g–CALCULUS

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Abstract

In this paper some theorems of g–calculus analogous to the theorems of clasical calculus are proved. Among others: theorem on derivative of inverse function, Leibniz formula, partial integration and mean value theorem for g–integral.

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1. Introduction

The notion of g–calculus was introduced by E. Pap [4]. It is based on the operations of pseudo - addition and pseudo -multiplication ([1], [2], [5], [7], [8]) and it is a generalization of the classical calculus. It is very usefull in the theory of nonlinear differential equations (see [4], [5]).

In this paper we shall prove some theorems which are analogous to the classical theorems of the usual calculus.

In the second section we shall give some neccesery notions and notations from g–calculus, based on [4]. In section 3 we shall prove the Leibniz formula for g–derivative. In section 4 we shall prove the analogues of theorems on partial integration and the mean value theorem. In the fifth and sixth section we shall give the tables of g–derivatives and g–integrals, respectively, for some generators g.

2. Preliminary notions and notations

Let $[a,b]$ be a closed real interval. The operation $\oplus$ (pseudo - addition) is a function $\oplus : [a,b]\times[a,b]\to[a,b]$ which is commutative, nondecreasing, associative and has a zero element, denoted by $\mathbf{0}$ (which is either a or b). We shal consider only strict pseudo - addition, i.e. such that the function $\oplus$ is continuous and strictly increasing in $(a,b)\times(a,b)$.

By Aczel's theorem for each strict pseudo - addition $\oplus$ there exists a monotone function g (generator for $\oplus$), $g: [a,b]\to[0,\infty]$ such that either $g(a)=0$ or $g(b)=0$ and

$$x\oplus y = g^{-1}(g(x)+g(y)).$$

The operation $\otimes$ (pseudo - multiplication) is a function $\otimes : [a,b]\times[a,b]\to[a,b]$ which is commutative, nondecreasing, associative and has a unit element $\mathbf{1}$. We suppose further that the pseudo - multiplication is defined as

$$x\otimes y = g^{-1}(g(x)\cdot g(y))$$

where g is a generator of the strict pseudo - addition $\oplus$. $\otimes$ is distributive with respect to $\oplus$.

It is easily shown $g(\mathbf{0})=0$ and $g(\mathbf{1})=1$.

Lemma 1. *We mark by x^* the inverse element for element $x\in(a,b)$, with respect to the pseudo - multiplication $\otimes$. Then we have:*

$$x^* = g^{-1}\left(\frac{1}{g(x)}\right).$$

Proof. From $x\otimes x^* = \mathbf{1}$ is $g^{-1}(g(x)\cdot g(x^*)) = \mathbf{1}$, i.e. $g(x)\cdot g(x^*) = g(\mathbf{1}) = 1$. Since it is only for $x=\mathbf{0}$, $g(x)=0$, that is $g(x)>0$ for $x\in(a,b)$, then

$$x^* = g^{-1}\left(\frac{1}{g(x)}\right)\in(a,b). \quad \square$$

Let f_1 and f_2 be two functions defined on the interval $[c,d]\subset R\cup\{\pm\infty\}$ and with values in $[a,b]$. Then, we define for any $x\in[c,d]$

$$(f_1\oplus f_2)(x) = f_1(x)\oplus f_2(x)$$

$$(f_1 \otimes f_2)(x) = f_1(x) \otimes f_2(x)$$

and for any $\lambda \in [a, b]$

$$(\lambda \otimes f_1)(x) = \lambda \otimes f_1(x) .$$

3. g−derivative

Let g be continuously differentiable on (a, b). We have from [4]:

Definition 1. *Let the function f be defined on the interval $[c, d]$ and with values in $[a, b]$. If f is differentiable on (c, d), and has same monotonicity as the function g then we define the g−derivative of f at the point $x \in (c, d)$ as*

$$\frac{d^{\oplus} f(x)}{dx} := g^{-1}\left(\frac{d}{dx} g(f(x))\right) .$$

Example 1. Let $g(x) = e^{-\frac{x}{c}}$, $c > 0$, $x \in R$. Then we have

$$\frac{d^{\oplus} f}{dx} = f - c\ln(-f') + c\ln c,$$

where f is strictly monotone decreasing, defined for each $x \in [c, d]$.

Example 2. Let $g(x) = x^p$, $p > 0$, $x \geq 0$. Then we have

$$\frac{d^{\oplus} f}{dx} = p^{\frac{1}{p}} \cdot f^{\frac{p-1}{p}} \cdot (f')^{\frac{1}{p}},$$

where is f strictly monoton increasing, defined for $x \in [c, d]$, and with values in $[0, \infty]$.

In the next theorem we shall use the following notation: the value of function f at the point x_0 will be denoted by $f[x_0]$. So $\frac{d^{\oplus}}{dx} f[x_0]$ means the value of g−derivative of f at the point x_0. By this convention we have $\frac{d^{\oplus}}{dx}[x_0] := \frac{d^{\oplus}}{dx} i[x_0]$, where $i(x) = x$.

Let g be a strictly increasing function. Then we have

Theorem 1. *Let f, $f : [c, d] \to [a, b]$, be a strictly increasing function, wich has g−derivative for each $x \in (c, d)$. Then we have:*

$$\frac{d^{\oplus}}{dx} f^{-1}[y_0] = \frac{d^{\oplus}}{dx}[f^{-1}(y_0)] \otimes \left(\frac{d^{\oplus}}{dx}(g^{-1} \circ f)[f^{-1}(y_0)]\right)^*$$

Proof. By theorem 4 from [4], we have:

$$\frac{d^{\oplus}}{dx}F[x_0] = \frac{d^{\oplus}}{dx}[F(x_0)] \otimes \frac{d^{\oplus}}{dx}(g^{-1} \circ h)[f(x_0)] \otimes \frac{d^{\oplus}}{dx}(g^{-1} \circ f)[x_0] \,.$$

If we take: $F = g = g \circ f^{-1} \circ f$ i.e. $h = g \circ f^{-1}$, we get

$$\frac{d^{\oplus}}{dx}g[x_0] = \frac{d^{\oplus}}{dx}[g(x_0)] \otimes \frac{d^{\oplus}}{dx}f^{-1}[f(x_0)] \otimes \frac{d^{\oplus}}{dx}(g^{-1} \circ f)[x_0]$$

Taking $y_0 = f(x_0)$, the preceding equality obtains the following form,

(1)
$$\frac{d^{\oplus}}{dx}f^{-1}[y_0] = (\frac{d^{\oplus}}{dx}[g(x_0)])^* \otimes \frac{d^{\oplus}}{dx}g[x_0] \otimes$$
$$\otimes (\frac{d^{\oplus}}{dx}(g^{-1} \circ f)[x_0])^*.$$

We have

(2)
$$(\frac{d^{\oplus}}{dx}[g(x_0)])^* = g^{-1}(\frac{1}{g(g^{-1}(\frac{d}{dx}g[g(x_0)]))}) = g^{-1}(\frac{1}{g'[g(x_0)]})$$

and

(3)
$$(\frac{d^{\oplus}}{dx}(g^{-1} \circ f)[x_0])^* = g^{-1}(\frac{1}{g(g^{-1}(\frac{d}{dx}g(g^{-1}(f[x_0]))))}) =$$
$$= g^{-1}(\frac{1}{\frac{d}{dx}f[x_0]}) = g^{-1}(\frac{1}{f'[x_0]}) \,.$$

Equalities (2) and (3) implie in (1)

$$\frac{d^{\oplus}}{dx}f^{-1}[y_0] = g^{-1}(\frac{1}{g'[g(x_0)]} \cdot \frac{d}{dx}g(g[x_0]) \cdot \frac{1}{f'[x_0]}) =$$

$$= g^{-1}(\frac{1}{g'[g(x_0)]} \cdot g'[g(x_0)] \cdot g'[x_0] \cdot \frac{1}{f'[x_0]}) =$$

$$= g^{-1}(\frac{g'[x_0]}{f'[x_0]}) = g^{-1}(g(g^{-1}(g'[x_0])) \cdot g(g^{-1}(\frac{1}{g(g^{-1}(f'[x_0]))}))) =$$

$$= g^{-1}(g'[x_0]) \otimes (g^{-1}(f'[x_0]))^* =$$

$$= \frac{d^{\oplus}}{dx}[x_0] \otimes (g^{-1}(\frac{d}{dx}(g \circ g^{-1} \circ f)[x_0]))^* =$$

$$= \frac{d^{\oplus}}{dx}[f^{-1}(y_0)] \otimes (\frac{d^{\oplus}}{dx}(g^{-1} \circ f)[f^{-1}(y_0)])^* \quad \square$$

Example 3. Let $f(x) = x^n$ and $g(x) = x^p$, $p > 0$. Then we have

$$(g^{-1} \circ f)(x) = x^{\frac{n}{p}}$$

$$\frac{d^{\oplus}}{dx}(g^{-1} \circ f)(x) = n^{\frac{1}{p}} x^{\frac{n-1}{p}}$$

$$(\frac{d^{\oplus}}{dx}(g^{-1} \circ f)(x))^* = n^{-\frac{1}{p}} x^{\frac{1-n}{p}}$$

$$\frac{d^{\oplus}}{dx}[x] \otimes (\frac{d^{\oplus}}{dx}(g^{-1} \circ f)(x))^* = (\frac{p}{n})^{\frac{1}{p}} x^{\frac{p-n}{p}} \; .$$

Hence by Theorem 1 we obtain

$$\begin{aligned} \frac{d^{\oplus}}{dx} f^{-1}[y_0] &= \frac{d^{\oplus}}{dx}[f^{-1}(y_0)] \otimes (\frac{d^{\oplus}}{dx}(g^{-1} \circ f)[f^{-1}(y_0)])^* \\ &= (\frac{p}{n})^{\frac{1}{p}} (\sqrt[n]{y_0})^{\frac{p-n}{p}} = (\frac{p}{n})^{\frac{1}{p}} y_0^{\frac{p-n}{np}} \; . \end{aligned}$$

In the rest of the paper the functions f_1 and f_2 will be defined on the interval $[c, d]$ and with the values in $[a, b]$.

We have the following generalization of Leibniz formula.

Theorem 2. *If there exist n g derivatives of the functions f_1 and f_2, then we have*

$$\frac{d^{(n)\oplus}}{dx}(f_1 \otimes f_2) = \bigoplus_{k=0}^{n} g^{-1}(\binom{n}{k}) \otimes \frac{d^{(n-k)\oplus} f_1}{dx} \otimes \frac{d^{(k)\oplus} f_2}{dx} \; .$$

Proof. Using Theorem 3 from [4] and the classical Leibniz formula we obtain

$$\frac{d^{(n)\oplus}}{dx}(f_1 \otimes f_2) = g^{-1}(\frac{d^n}{dx^n} g(f_1 \otimes f_2))$$

$$= g^{-1}(\frac{d^n}{dx^n}(g(f_1) \cdot g(f_2)))$$

$$= g^{-1}(\sum_{k=0}^{n} \binom{n}{k} \frac{d^{(n-k)}}{dx^{n-k}}(g(f_1)) \cdot \frac{d^{(k)}}{dx^k}(g(f_2)))$$

$$= g^{-1}\{\sum_{k=0}^{n} g\{g^{-1}\{\binom{n}{k}\cdot g(g^{-1}(\frac{d^{(n-k)}}{dx^{n-k}}(g(f_1))))\cdot g(g^{-1}(\frac{d^{(k)}}{dx^k}(g(f_2))))\}\}\} =$$

$$= \bigoplus_{k=0}^{n} g^{-1}\{g(g^{-1}(\binom{n}{k}))\cdot g(\frac{d^{(n-k)\oplus}f_1}{dx})\cdot g(\frac{d^{(k)\oplus}f_2}{dx})\} =$$

$$= \bigoplus_{k=0}^{n} g^{-1}(\binom{n}{k})\otimes \frac{d^{(n-k)\oplus}f_1}{dx}\otimes\frac{d^{(k)\oplus}f_2}{dx}. \quad \square$$

4. $g-$ integral

We have from [4]:

Definition 2. *Let the function f be defined on the interval $[c,d]$ and with values in $[a,b]$. If f is measurable on $[c,d]$ we define*

$$\int_{[c,d]}^{\oplus} f(x)dx := g^{-1}(\int_c^d g(f(x))dx) .$$

Theorem 3. *Let f_1 and f_2 be continuous $g-$diferentiable on the interval (c,d). Then for each $x\in(c,d)$ we have*

$$\int_{[c,x]}^{\oplus}(\frac{d^{\oplus}}{dx}f_1(x)\otimes f_2(x))dx \oplus \int_{[c,x]}^{\oplus}(f_1(x)\otimes\frac{d^{\oplus}}{dx}f_2(x))dx\oplus(f_1(c)\otimes f_2(c)) =$$

$$= f_1(x)\otimes f_2(x).$$

Proof. Theorems 2, 5 i 7 from [4] imply

$$\int_{[c,x]}^{\oplus}\frac{d^{\oplus}}{dx}(f_1(x)\otimes f_2(x))dx\oplus(f_1(c)\otimes f_2(c)) =$$

$$= \int_{[c,x]}^{\oplus}(\frac{d^{\oplus}}{dx}f_1(x)\otimes f_2(x)\oplus f_1(x)\otimes\frac{d^{\oplus}}{dx}f_2(x))dx\oplus(f_1(c)\otimes f_2(c)),$$

i.e.

$$f_1(x)\otimes f_2(x) = \int_{[c,x]}^{\oplus}(\frac{d^{\oplus}}{dx}f_1(x)\otimes f_2(x))dx\oplus\int_{[c,x]}^{\oplus}(f_1(x)\otimes\frac{d^{\oplus}}{dx}f_2(x))dx\oplus (f_1(c)\otimes f_2(c)). \quad \square$$

Theorem 4. *Let f_1, f_2 be two measurable functions on $[c,d]$. If the function f_1 is continuous on the interval $[c,d]$, then there exists $\xi \in [c,d]$ such that*

$$\int_{[c,d]}^{\oplus} f_1(x) \otimes f_2(x)dx = f_1(\xi) \otimes \int_{[c,d]}^{\oplus} f_2(x)dx$$

Proof. Since the composition of continuous functions and measurable function is a measurable function, there exists the integral on the left side of the following equality

$$\begin{aligned}\int_{[c,d]}^{\oplus} f_1(x) \otimes f_2(x)dx &= g^{-1}(\int_c^d g(g^{-1}(g(f_1(x)) \cdot g(f_2(x))))dx) \\ &= g^{-1}(\int_c^d g(f_1(x)) \cdot g(f_2(x))dx)\,.\end{aligned}$$

Since the composition $g \circ f_1$ is continuous on $[c,d]$ and $g(f_2(x)) \geq 0$, for each $x \in [c,d]$, by the theorem of mean value, there exists $\xi \in [c,d]$, such that

$$\begin{aligned}\int_{[c,d]}^{\oplus} f_1(x) \otimes f_2(x)dx &= g^{-1}(g(f_1(\xi)) \cdot \int_c^d g(f_2(x))dx) \\ &= g^{-1}(g(f_1(\xi)) \cdot g(g^{-1}(\int_c^d g(f_2(x))dx))) \\ &= f_1(\xi) \otimes \int_{[c,d]}^{\oplus} f_2(x)dx\,,\end{aligned}$$

i.e. the assertion is valid. □

Example 4. Let f_1 and f_2, $f_1 : [c,d] \to [-\infty,\infty]$, $f_2 : [c,d] \to [-\infty,\infty]$ be two measurable functions on $[c,d]$, and f_1 a continuous function. For $g_2(x) = e^{-\frac{x}{k}}$, $k > 0$, we have:

$$\int_{[c,d]}^{\oplus} f_1(x) \otimes f_2(x)dx = -k\ln(\int_c^d e^{-\frac{f_1(x)+f_2(x)}{k}}dx)$$

$$f_1(\xi) \otimes \int_{[c,d]}^{\oplus} f_2(x)dx = f_1(\xi) - k\ln(\int_c^d e^{-\frac{f_2(x)}{k}}dx), \quad \xi \in [c,d].$$

5. Table of $g-$ derivatives

The following table represents the ordinary derivative and the corresponing $g-$derivatives for $g_1(x) = x^p, g_1 : [0,\infty] \to [0,\infty]$, $p > 0$ and $g_2(x) = e^{-\frac{x}{c}}, g_2 : [-\infty,\infty] \to [0,\infty]$, $c > 0$ of some elementary functions.

Let us introduce the notation $f_i'(x) = \frac{d^{\oplus} f}{dx} = g_i^{-1}(\frac{d}{dx} g(f(x)))$, $i = 1,2$.

(1)
$$f(x) = a,\ x \in R, a \in R;$$
$$f'(x) = 0;$$
$$f_1'(x) = 0,\ a \geq 0;$$
$$\rule{3cm}{0.4pt}.$$

(2)
$$f(x) = x^\alpha,\ x > 0, \alpha \in R;$$
$$f'(x) = \alpha x^{\alpha-1};$$
$$f_1' x = (\alpha p)^{\frac{1}{p}} x^{\frac{\alpha p-1}{p}}, x > 0, \alpha \geq 0;$$
$$\rule{3cm}{0.4pt}.$$

(3)
$$f(x) = x^\alpha,\ x \neq 0, \alpha = \frac{m}{n} \in Q, n - \text{odd};$$
$$f'(x) = \alpha x^{\alpha-1};$$
$$f_1'(x) = (\alpha p)^{\frac{1}{p}} x^{\frac{\alpha p-1}{p}}, x > 0, \alpha \geq 0;$$
$$f_2'(x) = x^\alpha - c \ln\frac{\alpha}{c} - c\ln(-x^{\alpha-1}),$$
$$x < 0, \alpha = \frac{m}{n} > 0, m - \text{even}, n - \text{odd}.$$

(4)
$$f(x) = e^x,\ x \in R;$$
$$f'(x) = e^x;$$
$$f_1'(x) = p^{\frac{1}{p}} e^x;$$
$$\rule{3cm}{0.4pt}.$$

(5)
$$f(x) = a^x,\ x \in R, a > 0, a \neq 1;$$
$$f'(x) = a^x \ln a;$$
$$f_1'(x) = (p \ln a)^{\frac{1}{p}} \cdot a^x, x \in R, a > 1;$$
$$f_2'(x) = a^x - cx \ln a + c \ln(-\frac{c}{\ln a}), x \in R, 0 < a < 1.$$

(6)
$$f(x) = \ln x,\ x > 0;$$
$$f'(x) = \frac{1}{x};$$
$$f_1'(x) = (\frac{p}{x})^{\frac{1}{p}} \cdot (\ln x)^{\frac{p-1}{p}}, x > 1;$$

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(7)
$$f(x) = \log_a x,\ x > 0, a > 0, a \neq 1;$$
$$f'(x) = \frac{1}{x \ln a};$$
$$f_1'(x) = \frac{p^{\frac{1}{p}}}{\ln a} \cdot (\frac{(\ln x)^{p-1}}{x})^{\frac{1}{p}}, x > 1, a > 1;$$
$$f_2'(x) = \log_a x + c \ln x + c \ln(-c \ln a), x > 0, 0 < a < 1.$$

(8)
$$f(x) = \sin x,\ x \in R;$$
$$f'(x) = \cos x;$$
$$f_1'(x) = (p \sin^{p-1} x \cdot \cos x)^{\frac{1}{p}}, x \in (2k\pi, \frac{\pi}{2} + 2k\pi], k \in Z;$$
$$f_2'(x) = \sin x + c \ln c - c \ln(-\cos x), x \in (\frac{\pi}{2} + 2k\pi, \frac{3\pi}{2} + 2k\pi), k \in Z.$$

(9)
$$f(x) = \cos x,\ x \in R;$$
$$f'(x) = -\sin x;$$
$$f_1'(x) = (-p \cos^{p-1} x \sin x)^{\frac{1}{p}}, x \in (-\frac{\pi}{2} + 2k\pi, 2k\pi], k \in Z;$$
$$f_2'(x) = \cos x + c \ln c - c \ln(\sin x), x \in (2k\pi, (2k+1)\pi), k \in Z.$$

(10)
$$f(x) = \mathrm{tg}x,\ x \neq (2k+1)\frac{\pi}{2}, k \in Z;$$
$$f'(x) = \frac{1}{\cos^2 x};$$
$$f_1'(x) = p^{\frac{1}{p}} \cdot \frac{\mathrm{tg}x}{(\sin x \cos x)^{\frac{1}{p}}}, x \in (k\pi, \frac{\pi}{2} + k\pi), k \in Z;$$
$$\underline{\qquad\qquad}.$$

(11)
$$f(x) = \mathrm{ctg}x,\ x \neq k\pi, k \in Z;$$
$$f'(x) = -\frac{1}{\sin^2 x};$$
$$\underline{\qquad\qquad};$$
$$f_2'(x) = \mathrm{ctg}x + c\ln c + c\ln(\sin^2 x).$$

(12)
$$f(x) = \arcsin x,\ |x| \leq 1;$$
$$f'(x) = \frac{1}{\sqrt{1-x^2}}, |x| < 1;$$
$$f_1'(x) = p^{\frac{1}{p}}(\arcsin x)^{\frac{p-1}{p}} \cdot (1-x^2)^{-\frac{1}{2p}}, x \in (0,1);$$
$$\underline{\qquad\qquad}.$$

(13)
$$f(x) = \arccos x,\ |x| \leq 1;$$
$$f'(x) = -\frac{1}{\sqrt{1-x^2}}, |x| < 1;$$
$$\underline{\qquad\qquad};$$
$$f_2'(x) = \arccos x + c\ln c + \frac{c}{2}\ln(1-x^2), |x| < 1.$$

(14)
$$f(x) = \mathrm{arctg}x;\ x \in R;$$
$$f'(x) = \frac{1}{1+x^2};$$
$$f_1'(x) = p^{\frac{1}{p}}(\mathrm{arctg}x)^{\frac{p-1}{p}} \cdot (1+x^2)^{-\frac{1}{p}}, x > 0;$$
$$\underline{\qquad\qquad}.$$

(15)
$$f(x) = \operatorname{arcctg} x,\ x \in R;$$
$$f'(x) = -\frac{1}{1+x^2};$$
$$\underline{\qquad\qquad};$$
$$f_2'(x) = \operatorname{arcctg} x + c\ln c + c\ln(1+x^2).$$

(16)
$$f(x) = \operatorname{sh} x,\ x \in R;$$
$$f'(x) = \operatorname{ch} x;$$
$$f_1'(x) = p^{\frac{1}{p}} \operatorname{sh} x (\operatorname{cth} x)^{\frac{1}{p}}, x > 0;$$
$$\underline{\qquad\qquad}.$$

(17)
$$f(x) = \operatorname{ch} x,\ x \in R;$$
$$f'(x) = \operatorname{sh} x;$$
$$f_1'(x) = p^{\frac{1}{p}} \operatorname{ch} x (\operatorname{th} x)^{\frac{1}{p}}, x \geq 0;$$
$$f_2'(x) = \operatorname{ch} x + c\ln c - c\ln(-\operatorname{sh} x), x < 0.$$

(18)
$$f(x) = \operatorname{th} x,\ x \in R;$$
$$f'(x) = \frac{1}{\operatorname{ch}^2 x};$$
$$f_1'(x) = p^{\frac{1}{p}} \operatorname{th} x (\operatorname{sh} x \operatorname{ch} x)^{-\frac{1}{p}}, x > 0;$$
$$\underline{\qquad\qquad}.$$

(19)
$$f(x) = \operatorname{cth} x,\ x \neq 0;$$
$$f'(x) = -\frac{1}{\operatorname{sh}^2 x};$$
$$\underline{\qquad\qquad};$$
$$f_2'(x) = \operatorname{cth} x + c\ln c + c\ln(\operatorname{sh}^2 x).$$

The corresponding domain for the function is given either in the same box where the function is or in the previous box.

If the g−derivative does not exist, we write ________.

6. Table of $g-$ integrals

The following table represents the ordinary integral and the corresponing g−integrals for $g_1(x) = x^p, g_1 : [0,\infty] \to [0,\infty]$, $p > 0$ and $g_2(x) = e^{-\frac{x}{c}}, g_2 : [-\infty,\infty] \to [0,\infty]$, $c > 0$ of some elementary functions.

Let us introduce the notation $F_i(x) = \int^{\oplus} f(x)dx = g_i^{-1}(\int g(f(x))dx)$, $i = 1, 2$.

The constant C in the table is arbitrary but for the expressions have meaning.

(1)

$$f(x) = a,\ x \in R, a \in R;$$
$$\int f(x)dx = C;$$
$$F_1(x) = (a^p x + C)^{\frac{1}{p}}, x \in R, a \geq 0;$$
$$F_2(x) = -c\ln(xe^{-\frac{a}{c}} + C), x \in R.$$

(2)

$$f(x) = x^{\alpha},\ x > 0, \alpha \in R;$$
$$\int f(x)dx = \frac{x^{\alpha+1}}{\alpha+1} + C, x > 0, \alpha \neq -1;$$
$$\alpha p \neq -1,\ F_1(x) = (\frac{x^{\alpha p+1}}{\alpha p+1} + C)^{\frac{1}{p}}, x > 0;$$
$$F_2(x) = -c\ln(-2c\frac{\sqrt{x}+c}{e^{\sqrt{x}/c}} + C), \alpha = \frac{1}{2}, x > 0.$$

(3)
$$f(x) = \frac{1}{x},\ x \neq 0;$$
$$\int f(x)dx = \ln|x| + C;$$
$$p \neq 1, F_1(x) = (\frac{x^{-p+1}}{-p+1} + C)^{\frac{1}{p}}, x > 0;$$
$*.$

(4)
$$f(x) = e^x,\ x \in R;$$
$$\int f(x)dx = e^x + C;$$
$$F_1(x) = (\frac{1}{p}e^{px} + C)^{\frac{1}{p}};$$
$*.$

(5)
$$f(x) = a^x,\ x \in R, a > 0, a \neq 1;$$
$$\int f(x)dx = \frac{a^x}{\ln a} + C;$$
$$F_1(x) = (\frac{a^{px}}{p\ln a} + C)^{\frac{1}{p}};$$
$*.$

(6)
$$f(x) = \ln x,\ x > 0;$$
$$\int f(x)dx = x(\ln x - 1) + C;$$
$$p = 2,\ F_1(x) = (x\ln^2 x - 2x\ln x + 2x + c)^{\frac{1}{2}}, x \geq 1,\ **;$$
$$c \neq 1, F_2(x) = -c\ln(\frac{x^{1-1/c}}{1-1/c} + C), x > 0.$$

(7)
$$f(x) = \log_a x,\ x > 0, a > 0, a \neq 1;$$
$$\int f(x)dx = \frac{x}{\ln a}(\ln x - 1) + C;$$
$$p = 2,\ F_1(x) = (\frac{1}{\ln^2 a}(x\ln^2 x - 2x\ln x + 2x) + C)^{\frac{1}{2}},$$
$$0 < a < 1, x \in (0, 1], \text{or} a > 1, x \geq 1,\ **;$$
$$c = lna \neq 1, F_2(x) = -c\ln(\frac{x^{1-\frac{1}{c\ln a}}}{1 - \frac{1}{c\ln a}} + C), x > 0.$$

(8)
$$f(x) = \sin x,\ x \in R;$$
$$\int f(x)dx = -\cos x + C;$$
$$p = 2,\ F_1(x) = (\frac{x}{2} - \frac{\sin 2x}{4} + C)^{\frac{1}{2}},$$
$$x \in [2k\pi, (2k+1)\pi], k \in Z,\ **;$$
$$*.$$

(9)

(10)
$$f(x) = \cos, x\ x \in R;$$
$$\int f(x)dx = \sin x + C;$$
$$p = 2,\ F_1(x) = (\frac{x}{2} + \frac{\sin 2x}{4} + C)^{\frac{1}{2}},$$
$$x \in [-\frac{\pi}{2} + 2k\pi, \frac{\pi}{2} + 2k\pi], k \in Z,\ **;$$
$$*.$$

(11)
$$f(x) = \text{tg}x\ x \neq (2k+1)\frac{\pi}{2}, k \in Z;$$
$$\int f(x)dx = -\ln|\cos x| + C;$$
$$p = 2,\ F_1(x) = (-x + \text{tg}x + C)^{\frac{1}{2}}, x \in [2k\pi,$$

$\frac{\pi}{2} + 2k\pi), k \in Z,$ **;

*.

(12)
$$f(x) = \text{ctg}x, \ x \neq k\pi, k \in Z;$$
$$\int f(x)dx = \ln|\sin x| + C;$$
$$p = 2, \ F_1(x) = (-x - \text{ctg}x + C)^{\frac{1}{2}},$$
$$x \in (2k\pi, \frac{\pi}{2} + 2k\pi], \ **;$$
*.

(13)
$$f(x) = \frac{1}{\cos^2 x}, \ x \neq (2k+1)\frac{\pi}{2}, k \in Z;$$
$$\int f(x)dx = \text{tg}x + C;$$
$$p = 2, \ F_1(x) = (\frac{\sin x}{3\cos^3 x} + \frac{2\sin x}{3\cos x} + C)^{\frac{1}{2}},$$
$$p = \frac{1}{2}, \ F_1(x) = (\ln\left|\frac{1+\sin x}{\cos x}\right| + C)^2, \ **;$$
*.

(14)
$$f(x) = \frac{1}{\sin^2 x}, \ x \neq k\pi, k \in Z;$$
$$\int f(x)dx = -\text{ctg}x + C;$$
$$p = 2, \ F_1(x) = (\frac{-\cos x}{3\sin^3 x} - \frac{2\cos x}{3\sin x} + C)^{\frac{1}{2}},$$
$$p = \frac{1}{2}, \ F_1(x) = (\ln\left|\frac{1-\cos x}{\sin x}\right| + C)^2, \ **;$$
*.

(15) $f(x) = \arcsin x, \ |x| \leq 1;$

$$\int f(x)dx = x\arcsin x + \sqrt{1-x^2} + C;$$

$p = 2,\ F_1(x) = (-2x + 2\sqrt{1-x^2}\arcsin x + x\arcsin^2 x + C)^{\frac{1}{2}}, x \in [0,1];$

*.

(16) $f(x) = \arccos x,\ |x| \leq 1;$

$$\int f(x)dx = x\arccos x - \sqrt{1-x^2} + C;$$

$p = 2,\ F_1(x) = (-2x - 2\sqrt{1-x^2}\arccos x + x\arccos^2 x + C)^{\frac{1}{2}}, |x| \leq 1;$

*.

(17) $f(x) = \mathrm{sh}x,\ x \in R;$

$$\int f(x)dx = \mathrm{ch}x + C;$$

$p = 2,\ F_1(x) = (\frac{\mathrm{sh}2x}{4} - \frac{x}{2} + C)^{\frac{1}{2}}, x \geq 0,$ **;

*.

(18) $f(x) = \mathrm{ch}x,\ x \in R;$

$$\int f(x)dx = \mathrm{sh}x + C;$$

$p = 2,\ F_1(x) = (\frac{\mathrm{sh}2x}{4} + \frac{x}{2} + C)^{\frac{1}{2}},$ **;

*.

(19) $f(x) = \mathrm{tgh}x,\ x \in R;$

$$\int f(x)dx = \ln(\mathrm{ch}x) + C;$$

$p = 2,\ F_1(x) = (\frac{-2}{1+e^{-2x}} + x + C)^{\frac{1}{2}}, x \geq 0,$ **;

*.

(20)
$$f(x) = \mathrm{cth}x, \ x \neq 0;$$
$$\int f(x)dx = -\ln|\mathrm{sh}x| + C;$$
$$p = 2, \ F_1(x) = (\frac{2}{-1+e^{-2x}} + x + C)^{\frac{1}{2}}, x > 0, \ **;$$
*.

(21)
$$f(x) = \frac{1}{\sqrt{1-x^2}}, \ x \in (-1,1);$$
$$\int f(x)dx = \arcsin x + C;$$
$$p = 2, \ F_1(x) = (\arcsin x + C)^{\frac{1}{2}},$$
$$p = 3, \ F_1(x) = (x(1-x^2)^{-\frac{1}{2}} + C)^{\frac{1}{3}}, \ **;$$
*.

(22)
$$f(x) = \frac{1}{1+x^2}, \ x \in R;$$
$$\int f(x)dx = \mathrm{arctg}x + C;$$
$$p = 2, \ F_1(x) = (\frac{x}{2(1+x^2)} + \frac{\mathrm{arctg}x}{2} + C)^{\frac{1}{2}},$$
$$p = \frac{1}{2}, \ F_1(x) = (\ln|x + \sqrt{1+x^2}| + C)^2, \ **;$$
*.

(23)
$$f(x) = \frac{1}{\sqrt{x^2+1}}, \ x \in R;$$
$$\int f(x)dx = \ln|x + \sqrt{x^2+1}| + C;$$
$$p = 2, \ F_1(x) = (\mathrm{arctg}x + C)^{\frac{1}{2}},$$
$$p = 3, \ F_1(x) = (x(1+x^2)^{-\frac{1}{2}} + C)^{\frac{1}{3}}, \ **;$$
*.

(24)
$$f(x) = \frac{1}{\sqrt{x^2-1}}, \ |x| > 1;$$
$$\int f(x)dx = \ln|x + \sqrt{x^2-1}| + C;$$
$$p = 2, \ F_1(x) = (-\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + C)^{\frac{1}{2}},$$
$$p = 3, \ F_1(x) = (-x(x^2-1)^{-\frac{1}{2}} + C)^{\frac{1}{3}}, \ **;$$
*.

(25)
$$f(x) = \frac{1}{1-x^2}, \ x \neq \pm 1;$$
$$\int f(x)dx = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + C;$$
$$p = \frac{1}{2}, \ F_1(x) = (\arcsin x + C)^2,$$
$$p = 2, \ F_1(x) = (\frac{x}{2(1-x^2)} - \frac{1}{4}\ln\left|\frac{1+x}{1-x}\right| + C)^{\frac{1}{2}}, x \in (-1,1), \ **;$$
*.

The mark * means that the integral is not calculated.

The mark ** means that we can calculate the g−integrate for natural number.

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REZIME

NEKE NOVE OSOBINE g−RAČUNA

U ovom radu se dokazuju neke teoreme g−računa, analogne teoremama klasičnog diferencijalnog i integralnog računa: teorema o izvodu inverzne funkcije, teorema o srednjoj vrednosti g− integrala i druge teoreme.

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