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ON THE OVERRELAXATION METHOD WITH SEVERAL RELAXATION PARAMETERS

Ljiljana Cvetković, Dragoslav Herceg, Nenad Petrović
Institute of Mathematics, University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

There are a lot multiparameter relaxation methods for solving systems of linear equations. In this paper we consider a threeparameter relaxation method proposed by Moussavi in [10] and prove the convergence area for the class of H-matrices. As special cases of this method appear the AOR (Accelerated Overrelaxation) and MSOR (Modified Successive Overrelaxation) method.

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1. Introduction

Well - known methods for solving systems of linear equations

$$Ax = b, \tag{1}$$

are SOR (Successive Overrelaxation) and AOR (Accelerated Overrelaxation) method. Both of them can be considered as methods obtained by a splitting of Varga's type.

For SOR method we have

$$M = \frac{1}{\omega}(D - \omega T),\ N = \frac{1}{\omega}\left((1-\omega)D + \omega S\right),$$

and for AOR method

$$M = \frac{1}{\omega}(D - \sigma T),\ N = \frac{1}{\omega}\left((1-\omega)D + (\omega - \sigma)T + \omega S\right),$$

where $A = D - T - S$. Here, D is a diagonal matrix and T, S are strictly lower and strictly upper triangular matrices, respectively. Also, we assume that the matrix A has nonzero diagonal elements and σ, ω are real parameters, $\omega \neq 0$.

Since $A = M - N$, for arbitrary x^0 we can define the iterations in a following way:

$$Mx^{k+1} - Nx^k = b,\quad k = 0, 1, 2, \ldots,$$

or equivalently

$$x^{k+1} = M^{-1}Nx^k + M^{-1}b,\quad k = 0, 1, 2, \ldots.$$

The matrix $M^{-1}N$ is called the iteration matrix and

$$\mathcal{L}_\omega = (D - \omega T)^{-1}\left((1-\omega)D + \omega S\right)$$

and

$$\mathcal{M}_{\sigma,\omega} = (D - \sigma T)^{-1}\left((1-\omega)D + (\omega - \sigma)T + \omega S\right)$$

are the SOR and the AOR iterative matrices respectively.

For the special linear systems there are the special iterative methods. One of them is MSOR (Modified Successive Overrelaxation) method, introduced by Devogelaere [3]. The matrix A has the following form:

$$A = \begin{bmatrix} D_1 & M \\ N & D_2 \end{bmatrix}, \tag{2}$$

where D_1 and D_2 are square nonsingular matrices. Use ω for the "red" equations corresponding to D_1 and $\bar{\omega}$ for the "black" equations corresponding to D_2 and

$$M = \begin{bmatrix} \frac{1}{\omega}D_1 & 0 \\ N & \frac{1}{\bar{\omega}}D_2 \end{bmatrix},$$

$$N = M - A = \begin{bmatrix} (\frac{1}{\omega} - 1)D_1 & -M \\ 0 & (\frac{1}{\bar{\omega}} - 1)D_2 \end{bmatrix},$$

we obtain MSOR iterative matrix:

$$\mathcal{H}_{\omega,\bar{\omega}} = M^{-1}N = \begin{bmatrix} (1-\omega)E_1 & \omega F \\ \bar{\omega}(1-\omega)G & \omega\bar{\omega}GF + (1-\bar{\omega})E_2 \end{bmatrix},$$

where $F = -D_1^{-1}M$, $G = -D_2^{-1}N$ and E_1, E_2 are identity matrices.

Some convergence results for the MSOR method are given in [14], [4], [12], [8], [9], [1].

2. Method to be considered

Here we shall investigate three parameter SOR method proposed by Moussavi in [10]. The iteration matrix for this method is

$$(3)\quad \mathcal{B}_{\omega,\bar{\omega},\alpha} = (E - \alpha\bar{\omega}Q)^{-1}((1-\omega)E + \omega B + \bar{\omega}(1-\alpha)Q + (\omega - \bar{\omega})R),$$

where

$$Q = \begin{bmatrix} 0 & 0 \\ G & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}, R = \begin{bmatrix} 0 & 0 \\ 0 & E_2 \end{bmatrix},$$

and $F = -D_1^{-1}M$, $G = -D_2^{-1}N$.

Special cases of this method are:

- $\alpha = 1$, $\omega = \bar{\omega}$ SOR method,
- $\alpha = 1$, $\omega \neq \bar{\omega}$ MSOR method,
- $\alpha = \frac{\sigma}{\omega}$, $\omega = \bar{\omega}$ AOR method.

3 . Convergence results

Let us introduce the following notation:

$$P_i(A) = \sum_{j=1, j\neq i}^{n} |a_{ij}|, \quad i = 1, 2, \ldots, n, \quad A = [a_{ij}] \in \mathcal{C}^{n,n},$$

$$b_i = P_i(B), \quad q_i = P_i(Q), \quad r_i = P_i(R).$$

Theorem 1. *Let* $1 - |\alpha\bar{\omega}|q_i > 0, \;\; i = 1, 2, \ldots, n$. *Then*

$$\rho(\mathcal{B}_{\omega,\bar{\omega},\alpha}) \leq \max_{1 \leq i \leq n} \frac{|1 - \omega + (\omega - \bar{\omega})r_i| + |\bar{\omega}(1 - \alpha)|q_i + |\omega|b_i}{1 - |\alpha\bar{\omega}|q_i}$$

Proof. Suppose there exists an eigenvalue λ of the matrix $\mathcal{B}_{\omega,\bar{\omega},\alpha}$, such that

$$|\lambda| > \frac{|1 - \omega + (\omega - \bar{\omega})r_i| + |\bar{\omega}(1 - \alpha)|q_i + |\omega|b_i}{1 - |\alpha\bar{\omega}|q_i}, \quad i = 1, 2, \ldots n.$$

Then for each $i = 1, 2, \ldots n$, it follows that

$$|c_{ii}| > P_i(C), \quad i = 1, 2, \ldots n,$$

where

$$C = (\lambda - 1 + \omega)E - (\omega - \bar{\omega})R - \bar{\omega}(1 + \alpha(\lambda - 1))Q - \omega B.$$

It means that the matrix C is a regular matrix.

But, this is a contradiction with the fact that λ is an eigenvalue of the matrix $\mathcal{B}_{\omega,\bar{\omega},\alpha}$, i.e.

$$det(\lambda E - \mathcal{B}_{\omega,\bar{\omega},\alpha}) = det(E - \alpha\bar{\omega}Q)^{-1} det C = 0.$$

The proof is completed. $\square$

The previous theorem can be used for the convergence analysis of our three parameter SOR method. By analysing the inequality

$$\epsilon < 1,$$

where ϵ is the upper bound for spectral radius of the iterative matrix given in Theorem 1, the following statement can be proved.

Theorem 2. *Let A be a strictly diagonally dominant matrix of the form (2). If $b = \|B\|_\infty, \; q = \|Q\|_\infty$, then the method with iteration matrix (3) is convergent for:*

(i) $0 < \omega < \frac{2}{1+b}, \;\; 0 < \bar{\omega} \leq 1, \;\; -\frac{1-q}{2q} < \alpha < \frac{1+q}{2q}$ *or*

(ii) $0 < \omega < \frac{2}{1+b}, \;\; 1 \leq \bar{\omega} < \frac{2}{1+q}, \;\; -(\frac{1}{\bar{\omega}q} - \frac{1+q}{2q}) < \alpha < \frac{1}{\bar{\omega}q} - \frac{1-q}{2q}.$

Proof. If the relaxation parameters satisfy the condition (i) or (ii), then from Theorem 1 it follows that

$$\rho(\mathcal{B}_{\omega,\bar{\omega},\alpha}) < 1,$$

and convergence is proved. □

Now, we suppose that A is an H-matrix. Let W be a regular diagonal matrix, for which AW is strictly diagonally dominant. Such a matrix allways exists. Since

$$\mathcal{B}_{\omega,\bar{\omega},\alpha}(AW) = W^{-1}\mathcal{B}_{\omega,\bar{\omega},\alpha}(A)W,$$

the following theorem is valid.

Theorem 3. *Let A be an H-matrix of the form (2). If $b = \|W^{-1}BW\|_\infty$, $q = \|W^{-1}QW\|_\infty$, then the method with iteration matrix (3) is convergent for:*

$(i) 0 < \omega < \frac{2}{1+b},\ 0 < \bar{\omega} \leq 1,\ -\frac{1-q}{2q} < \alpha < \frac{1+q}{2q}$ *or*

$(ii) 0 < \omega < \frac{2}{1+b},\ 1 \leq \bar{\omega} < \frac{2}{1+q},\ -(\frac{1}{\bar{\omega}q} - \frac{1+q}{2q}) < \alpha < \frac{1}{\bar{\omega}q} - \frac{1-q}{2q}.$

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REZIME

O POSTUPKU GORNJE RELAKSACIJE SA VIŠE RELAKSACIONIH PARAMETARA

Postoji mnogo višeparametarskih relaksacionih postupaka za rešavanje sistema linearnih jednačina. U ovom radu razmatra se troparametarski relaksacioni postupak predložen u radu [10] i dokazuje oblast konvergencije za klasu H-matrica. Specijalni slučajevi ovog postupka su AOR (Accelerated Overrelaxation) i MSOR (Modified Successive Overrelaxation) postupak.

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