Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 23,2(1993), 401-410

Review of Research
Faculty of Science
Mathematics Series

ON A DISCRETE ANALOGUE FOR BOUNDARY VALUE PROBLEM

Dragoslav Herceg
University of Novi Sad, Institute of Mathematics
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

We consider a discrete analogue for 1-D boundary value problem arising by use a four-point difference scheme on nonuniform mesh. The matrix from this analogue we describe as a product two M-matrices and derive explicit inverse matrices.

AMS Mathematics Subject Classification (1991): 65L10, 65H10
Key words and phrases: discrete analogue, boundary value problem, finite differences, tridiagonal matrix, explicite inverse.

1. Introduction

Consider the following boundary value problem:

$$(1)\qquad \begin{aligned} -u'' = c(x,u), \quad x \in I = [0,1], \\ u(0) = a, \quad u(1) = b. \end{aligned}$$

We assume $f \in C(I \times \mathbf{R})$, and $a, b \in \mathbf{R}$. Futher assumptions will come into discussion later. We want to obtain a discrete solution of (1) on the mesh I_h defined by

$$(2)\qquad I_h = \{x_0 = 0, \quad x_i = x_{i-1} + hk_i, \quad i = 1, ..., n\},$$

where $n > 3, n \in \mathbf{N}, k_i > 0, i = 1, 2, ..., n, k_n = k_{n-1}$ and a mesh step

$$h^{-1} = \sum_{i=1}^{n} k_j.$$

If $k_j = 1,\ j = 1, \dots, n$ an uniform mesh is obtained and if there exist $i, j \in \{1, \dots, n\}$ such that $k_j \neq k_i$, I_h is a nonuniform mesh. Nonuniform meshes are commonly used for a numerical solution of singularly perturbed problems because of boundary layers, [1], [2], [8], [5], [6].

In both cases, using some approximations for $-u''(t)$ at each mesh point, we obtain discrete analogue which can be written as system of nonlinear equations. To obtain a discrete analogue we use four-point second order approximation for second derivative:

$$-u''(x_i) \approx h^{-2} \left(a_i u_{i-1} + b_i u_i + c_i u_{i+1} + d_i u_{i+2}\right) \tag{3}$$

where u_i is an approximation to $u(x_i)$ and for $i = 1, 2, ..., n-1$

$$a_i = \frac{-2(2k_{i+1} + k_{i+2})}{k_i(k_i + k_{i+1})(k_i + k_{i+1} + k_{i+2})}, \qquad b_i = \frac{2(-k_i + 2k_{i+1} + k_{i+2})}{k_i k_{i+1}(k_{i+1} + k_{i+2})},$$

$$c_i = \frac{2(k_i - k_{i+1} - k_{i+2})}{k_{i+1}(k_i + k_{i+1})k_{i+2}}, \qquad d_i = \frac{2(k_{i+1} - k_i)}{k_{i+2}(k_{i+1} + k_{i+2})(k_i + k_{i+1} + k_{i+2})}.$$

For $i = n-1$ we have $k_{i-1} = k_i$ and

$$a_{n-1} = \frac{-1}{k_n^2}, \qquad b_{n-1} = \frac{2}{k_n^2}, \qquad c_{n-1} = \frac{-1}{k_n^2}, \qquad d_{n-1} = 0.$$

The approximation (3) leads to the difference equation

$$h^{-2} \left(a_i u_{i-1} + b_i u_i + c_i u_{i+1} + d_i u_{i+2}\right) = f_i, \tag{4}$$

where $f_i = f(x_i, u_i)$.

Now, we obtain a numerical solution of the problem (1) as a solution of the nonlinear system

$$\begin{aligned} u_0 &= a, \\ h^{-2}(b_1u_1 + c_1u_2 + d_1u_3) &= f_1 - h^{-2}a_1a, \\ h^{-2}(a_iu_{i-1} + b_iu_i + c_iu_{i+1} + d_iu_{i+2}) &= f_i, \qquad i = 2,3,...,n-3, \\ h^{-2}(a_{n-2}u_{n-3} + b_{n-2}u_{n-2} + c_{n-2}u_{n-1}) &= f_{n-2} - h^{-2}d_{n-2}b, \\ h^{-2}(a_{n-1}u_{n-2} + b_{n-1}u_{n-1}) &= f_{n-1} - h^{-2}c_{n-1}b, \\ u_n &= b. \end{aligned}$$

This discrete analogue can be written in a matrix form

$$Au = F, \tag{5}$$

where $u = [u_1, u_2, ..., u_{n-1}]^\top$,

Matrix A is

$$A = h^{-2}\begin{bmatrix} b_1 & c_1 & d_1 & & & \\ a_2 & b_2 & c_2 & d_2 & & \\ & \cdots & \cdots & \cdots & & \\ & & a_{n-3} & b_{n-3} & c_{n-3} & d_{n-3} \\ & & & a_{n-2} & b_{n-2} & c_{n-2} \\ & & & & a_{n-1} & b_{n-1} \end{bmatrix},$$

and $F = [f_1, \ldots, f_{n-1}]^\top$.

More about numerical solution of the system (5) one can find in [5] and [6]. In [3] the same scheme is applied on special discretization mesh, and in [8] this scheme is used by numerical solution of singular perturbation problem. Our approximation (3) is, in general, second order, i.e.

$$-u''(x_i) = h^{-2}(a_iu_{i-1} + b_iu_i + c_iu_{i+1} + d_iu_{i+2}) + O(h^2), \tag{6}$$

if $u(x) \in C^4([0,1])$. More precisely

$$\begin{aligned} -u''(x_i) &= h^{-2}(a_iu_{i-1} + b_iu_i + c_iu_{i+1} + d_iu_{i+2}) - \\ &\frac{h^2}{12}(-k_ik_{i+1} + (k_{i+1} - k_i)(k_{i+1} + k_{i+2}))\,u^{(4)}(x_i) + O(h^3), \end{aligned} \tag{7}$$

for $u(x) \in C^5([0,1])$. It is obviously the approximation (3) is of order 3 if

$$\frac{1}{k_i} = \frac{1}{k_{i+1}} + \frac{1}{k_{i+1} + k_{i+2}}, \qquad i = 1, 2, ..., n-2. \tag{8}$$

The solution of (8) is, for some real constant c,

$$k_i = c\sqrt{2}^{\,i-1}, \qquad i = 1, 2, ..., n.$$

It is obvious that $k_{n-1} \neq k_n$. To be able use our scheme we define

$$k_i = \sqrt{2}^{\,i-1}, \qquad i = 1, 2, ..., n-1, \qquad k_{n-1} = k_n.$$

So, in this case our approximation (3) is of order 3 for $i = 1, 2, ..., n-2$, and of second order for $i = n-1$. But, using the method from [2], third order convergence of our scheme can be proven. In equidistant case, i.e. if $k_i = 1, i = 1, 2, ..., n$, our approximation becomes well known form

$$-u''(x_i) = h^{-2}(-u_{i-1} + 2u_i - u_{i+1}) + O(h^2). \tag{9}$$

The purpose of this paper is to obtain explicite inverse of teh matrix A under assumptions

$$1 \leq k_i \leq k_{i+1}, \qquad , i = 1, 2, ..., n-1, \quad k_{n-1} = k_n, \tag{10}$$

and derive some properties of A^{-1}. More about adventuage if the explicite inverse is known, can be found in [4], [9], [10].

2. Explicite inverse of A

Let us consider the matrix A assuming (10). It is easy to see that

$$A = MN,$$

where

$$M = \begin{bmatrix} r_1 & s_1 & & & \\ & r_2 & s_2 & & \\ & & \cdots & \cdots & \\ & & & r_{n-2} & s_{n-2} \\ & & & & r_{n-1} \end{bmatrix},$$

$$N = 2h^{-2}\begin{bmatrix} B_1 & C_1 & & & & \\ A_2 & B_2 & C_2 & & & \\ & & \dots & \dots & \dots & \\ & & & A_{n-2} & B_{n-2} & C_{n-2} \\ & & & & A_{n-1} & B_{n-1} \end{bmatrix},$$

and

$$r_i = \frac{2k_{i+1} + k_{i+2}}{k_i + k_{i+1} + k_{i+2}}, \quad i = 1, \dots, n-2, \qquad r_{n-1} = 1,$$

$$s_i = \frac{k_i - k_{i+1}}{k_i + k_{i+1} + k_{i+2}}, \quad i = 1, \dots, n-2,$$

$$A_i = \frac{-1}{k_i(k_i + k_{i+1})}, \quad i = 2, 3, \dots, n-2, \qquad B_i = \frac{1}{k_i k_{i+1}}, \quad i = 1, \dots, n-1,$$

$$C_i = \frac{-1}{k_{i+1}(k_{i+1} + k_i)}, \quad i = 1, \dots, n-2.$$

Theorem 1. *Matrix $P = M^{-1}$ is given by $P = [p_{ij}]$,*

$$p_{ij} = \begin{cases} 0, & i > j \\ \dfrac{k_j + k_{j+1} + k_{j+2}}{2k_{j+1} + k_{j+2}}, & i = j < n-1 \\ \dfrac{k_j + k_{j+1} + k_{j+2}}{2k_{j+1} + k_{j+2}} \displaystyle\prod_{m=i}^{j-1} \frac{k_{m+1} - k_m}{2k_{m+1} + k_{m+2}}, & i < j < n-1, \\ \displaystyle\prod_{m=i}^{n-2} \frac{k_{m+1} - k_m}{2k_{m+1} + k_{m+2}}, & i < n-1, \quad j = n-1, \\ 1, & i = j = n-1. \end{cases}$$

Proof. We have to prove equalities

$$r_i p_{ij} + s_i p_{i+1,j} = \delta_{ij}, \quad i = 1, \dots, n-1,$$

with

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

Since $p_{i,i} = 1/r_i$ and for $i < j$

$$p_{i,j} = \frac{k_{i+1} - k_i}{2k_{i+1} + k_{i+2}} p_{i,j} = \frac{-s_i}{r_i} p_{i+1,j},$$

we have

$$(M_2 M_2^{-1})_{i,i} = r_i r_i^{-1} + s_i \cdot 0 = 1,$$

and

$$r_i p_{ij} + s_i p_{i+1,j} = 0.$$

□

Theorem 2. *Matrix $Q = [q_{ij}] = N^{-1}$ is given by*

$$q_{ij} = \frac{1}{2} \begin{cases} (x_{j+1} - x_{j-1})(1 - x_j)x_i, & i \leq j, \\ (x_{j+1} - x_{j-1})(1 - x_i)x_j, & i \geq j. \end{cases}$$

Proof. With

$$h_i = hk_i, \qquad i = 1, 2, ..., n,$$

and

$$\alpha_i = \frac{h_{i+1}}{h_i}, \qquad \beta_i = \frac{2h_i}{h_{i+1} + h_i}, \qquad \gamma_i = \frac{1}{2h_i h_{i+1}},$$

we have

$$C_i = -\beta_i \gamma_i, \qquad B_i = \beta_i \frac{2}{\beta_i} \gamma_i, \qquad A_i = -\beta_i \alpha_i \gamma_i.$$

If we denote and

$$N_1 = 2h^{-2} diag(\gamma_1, \gamma_2, ..., \gamma_{n-1}),$$

and

$$N_2 = \begin{bmatrix} 2 & -\beta_1 & & & & \\ -\alpha_2\beta_2 & 2 & -\beta_2 & & & \\ & & \cdots & \cdots & \cdots & \\ & & & -\alpha_{n-2}\beta_{n-2} & 2 & -\beta_{n-2} \\ & & & & -\alpha_{n-1}\beta_{n-2} & 2 \end{bmatrix},$$

we obtain $N = N_1 N_2$ and $N^{-1} = N_2^{-1} N_1^{-1}$. Using result from [9] one can find

$$(N_2^{-1})_{ij} = \begin{cases} \dfrac{h_{j+1}+h_j}{2h_{j+1}h_j}(1-x_j)x_i, & i \le j, \\ \dfrac{h_{j+1}+h_j}{2h_{j+1}h_j}(1-x_i)x_j, & i \ge j, \end{cases}$$

or

$$(N_2^{-1})_{ij} = \frac{1}{h}\begin{cases} \dfrac{k_{j+1}+k_j}{2k_{j+1}k_j}(1-x_j)x_i, & i \le j, \\ \dfrac{k_{j+1}+k_j}{2k_{j+1}k_j}(1-x_i)x_j, & i \ge j, \end{cases}$$

Since

$$N_1^{-1} = diag(h_1h_2, h_2h_3, ..., h_{n-1}h_n) = h^2 diag(k_1k_2, k_2k_3, ..., k_{n-1}k_n),$$

it follows

$$(N^{-1})_{i,j} = \frac{h}{2}\begin{cases} (k_{j+1}+k_j)(1-x_j)x_i, & i \le j, \\ (k_{j+1}+k_j)(1-x_i)x_j, & i \ge j. \end{cases}$$

We end the proof with note $x_{j+1} - x_{j-1} = h(k_{j+1}+k_j)$.

□

As the corollary of Theorem1 and Theorem 2 we have

Theorem 3. *Matrix $A = [A_{ij}]$ is given by*

$$A_{i,j} = \frac{h^3}{2r_j}\begin{cases} \left(\sum\limits_{s=1}^{i} k_s\right)\sum\limits_{p=1}^{j}(k_{p+1}+k_p)\sum\limits_{s=p+1}^{n} k_s \prod\limits_{m=p}^{j-1}\dfrac{k_{m+1}-k_m}{2k_{m+1}+k_{m+2}}, & i \le j, \\ \left(\sum\limits_{s=i+1}^{n} k_s\right)\sum\limits_{p=1}^{j}(k_{p+1}+k_p)\sum\limits_{s=1}^{p} k_s \prod\limits_{m=p}^{j-1}\dfrac{k_{m+1}-k_m}{2k_{m+1}+k_{m+2}}, & i \ge j, \end{cases}$$

where for $p = j$ the product is to read as 1.

Proof. It is enough to remark that

$$p_{ij} = \frac{1}{r_j} \begin{cases} 0, & i > j \\ \prod_{m=i}^{j-1} \frac{-s_m}{r_m}, & i \leq j, \end{cases}$$

assuming for $i = j$

$$\prod_{m=i}^{j-1} \frac{-s_m}{r_m} = 1.$$

□

One can see that both of matrices M and N are L-matrices. Because of $M^{-1} = P \geq 0$ and $N^{-1} = Q \geq 0$ we conclude $A^{-1} = QP \geq 0$. Let $\delta = [1, 1, ..., 1]^\top$, then

$$P\delta = \delta, \qquad A^{-1}\delta = QP\delta = Q\delta,$$

and

$$\|A^{-1}\|_\infty = \|A^{-1}\delta\|_\infty = \|Q\delta\|_\infty = \|Q\|_\infty.$$

After simple calculation we find

$$\|Q\|_\infty = \max_{1 \leq i \leq n-1} \sum_{j=1}^{n-1} q_{i,j} = \frac{1}{2} \max_{1 \leq i \leq n-1} x_i(1 - x_i).$$

Since $x_i \in [0, 1]$, it follows

$$\|Q\|_\infty \leq \frac{1}{8}.$$

In equdistant case, i.e if $k_i = 1, \quad i = 1, 2, ..., n$, the matrix M is identity matrix and $A = N$. So,

$$\|A^{-1}\|_\infty \leq \frac{1}{8}.$$

References

[1] Bakhvalov,N.S.: K optimizacii metodov resheniya kraevykh zadach pri nalichii pogranichnogo sloya, Zh. vychisl. mat. i mat. fiz. 9 (4)(1969), 841-859.

[2] Bohl,E., Lorenz,J.: Inverse monotonicity and difference schemes of higher order. A summary for two-point boundary valu problems, Aeg. Math. 19(1979), 1-36.

[3] Cvetković,Lj., Herceg, D., On numerical solution of boundary value problem using an optimal numerical differentiation, Univ. u Novom Sadu, Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. 12(1982), 177-189.

[4] Fischer, C.F., Usmani, R.A., Properties of some tridiagonal matrices and their application to boundary value problems, SIAM J. Numer. Anal. 1(1969), 127-142.

[5] Herceg, D., Ein Differenzenverfahren zur Lösung von Randwertaufgaben, Univ. u Novom Sadu, Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. 9(1979), 221-232.

[6] Herceg, D., Nichtäquidistante Diskretisierung der Grenzschichtdifferentialglechungen und einige Eigenschaften von diskreten Analoga, Univ. u Novom Sadu, Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. 9(1979), 199-219.

[7] Herceg, D., The use of nonequdistant mesh in difference method. (Serbian. English summary), Univ. u Novom Sadu, Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. 10(1980), 102-112.

[8] Herceg,D., Vulanović,R.: Some finite-difference schemes for a singular perturbation problem on a non-uniform mesch, Univ u Novom Sadu Zb. Rad. Prrirod.-Mat. Fak. Ser. Mat. 11(1981), 117-134.

[9] Mattheij, R.M.M., Smooke, M.D., Estimations for the Inverse of Tridiagonal Matrices Arising in Boundary - Value Problems, Linear Algebra Appl. 73 (1986), 33 - 57.

[10] Usmani, R.A., Properties of a five-band matrix and its application to Boundary Value Problems, Linear Algebra Appl. 4(1971), 101-115.

REZIME

O JEDNOM DISKRETNOM ANALOGONU ZA KONTURNI PROBLEM

Posmatra se linearni deo diskretnog anologona konturnog problema dobijen pomoću četvorotačkaste diferencne aproksimacije drugog izvoda na proizvoljnoj neekvidistanoj mreži. Za dobijenu matricu odredjena je eksplicitno inverzna matrica i date su neke njene osobine.

Received by the editors April 12, 1993.