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HOW TO APPLY THEORY OF G-FUNCTIONS TO THE CONVERGENCE ANALYSIS OF THE RELAXATION METHODS

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Abstract

The concept of G-functions is used to obtain convergence area of the Accelerated Overrelaxation (AOR) method for solving some special linear systems. More precisely, a new upper bound for the spectral radius of the AOR iterative matrix is proved. The convergence result is a generalization of the statement from [7].

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1. Introduction

Let $\mathcal{C}^{n,n}$ and $\mathcal{C}^n$ denote the set of all $n \times n$ complex matrices and n-dimensional complex vectors, respectively. Let $\mathcal{P}_n, n \geq 2$, be the collection of all functions $f = [f_1, f_2, \ldots, f_n]^T$ such that for each $i = 1, 2, \ldots, n$, $f_i : \mathcal{C}^{n,n} \to \mathcal{R}$, $f_i(A) > 0$ for any $A \in \mathcal{C}^{n,n}$ and f_i depends only on the moduli of the off-diagonal entries of the matrices.

We begin with the definition given in [1].

Definition 1. *We say $f \in \mathcal{P}_n$ is a G-function if for each $A = [a_{ij}] \in \mathcal{C}^{n,n}$ satisfying*

$$(1) \qquad |a_{ii}| > f_i(A), \quad i = 1, 2, \ldots, n,$$

A is nonsingular.

We will denote by $\mathcal{G}_n$ the set of G-functions in $\mathcal{P}_n$. As examples, if

$$r_i(A) = \sum_{j=1, j \neq i}^{n} |a_{ij}|, \quad c_i(A) = \sum_{j=1, j \neq i}^{n} |a_{ji}|, \quad i = 1, 2, \ldots n,$$

then $r = [r_1, \ldots, r_n]^T$ and $c = [c_1, \ldots, c_n]^T$ are G-functions.

More generally, if $x = [x_1, \ldots, x_n]^T$ is any column vector in $\mathcal{C}^n$ with positive components ($x > 0$) and for $i = 1, 2, \ldots n$,

$$r_i^x(A) = \frac{1}{x_i} \sum_{j=1, j \neq i}^{n} |a_{ij}| x_j, \quad c_i^x(A) = \frac{1}{x_i} \sum_{j=1, j \neq i}^{n} |a_{ji}| x_j,$$

then $r^x = [r_1^x, \ldots, r_n^x]^T$ and $c^x = [c_1^x, \ldots, c_n^x]^T$ are G-functions.

The minimal G-functions are very important in our convergence analysis, so we shall present here the result from [1].

Given any reducible $A \in \mathcal{C}^{n,n}$, it is well known that there is a permutation matrix $P \in \mathcal{C}^{n,n}$, and a positive integer $m, 2 \leq m \leq n$, such that

$$(2) \qquad PAP^T = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \ldots & \tilde{A}_{1m} \\ 0 & \tilde{A}_{21} & \ldots & \tilde{A}_{2m} \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \tilde{A}_{mm} \end{bmatrix},$$

where each square submatrix $\tilde{A}_{kk}, k = 1, 2, \ldots m$, is either irreducible, or a 1×1 null matrix. The form (2) gives rise to a partitioning of $\{1, 2, \ldots n\}$ into m disjoint nonempty ssets $S_k = S_k(A)$, corresponding to the distinct connected components of the directed graph for A. The subsets S_k do not depend on the choice of the permutation matrix P which is used to obtain the form (2). For $i = 1, 2, \ldots n$, let $< i >$ denote the subset S_k containig i.

If $A \in \mathcal{C}^{n,n}$ is irreducible, we define $< i >= \{1, 2, \ldots n\}$ for all $i = 1, 2, \ldots n$.

We now define, for each $x \in C^n$ with $x > 0$ and $i = 1, 2, \dots n$,

$$(3) \qquad \hat{r}_i^x(A) = \frac{1}{x_i} \sum_{j \in <i>, j \neq i} |a_{ij}| x_j, \quad \hat{c}_i^x(A) = \frac{1}{x_i} \sum_{j \in <i>, j \neq i} |a_{ji}| x_j,$$

(we take $\hat{r}_i^x(A) = \hat{c}_i^x(A) = 0$ if $< i >= \{i\}$).

It is easy to see that $\hat{r}^x = [\hat{r}_1^x, \dots, \hat{r}_n^x]^T$ and $\hat{c}^x = [\hat{c}_1^x, \dots, \hat{c}_n^x]^T$ are G-functions and $\hat{r}^x \leq r^x$ and $\hat{c}^x \leq c^x$.

Theorem 1. *[[1]] Given any matrix $A \in C^{n,n}$ and $f \in \mathcal{G}_n$, there exists a vector $x \in C^n$ with $x > 0$ for which*

$$f_i(A) \geq \hat{r}_i^x(A), \quad i = 1, 2, \dots n.$$

2 . Method to be considered

The Accelerated Overrelaxation (AOR) method was introduced by Hadjidimos, [4], as a twoparameter generalization of the well known SOR method. In the paper [5] it is shown that in some special cases spectral radius of the iteration matrix can be zero, i.e. solution is obtained in at most n iterations (n is dimension of considered system) in exact arithmetics.

Let $A = D - T - S$ be the standard splitting of the matrix A into diagonal matrix D, strictly lower triangular matrix T and strictly upper triangular matrix S, and ω, σ are real parameters, $\omega \neq 0$. Also, we assume that the matrix A has nonzero diagonal elements.

The AOR method for solving system

$$(4) \qquad Ax = b,$$

is given with the following iteration rule

$$(5) (D - \sigma T) x^{k+1} = ((1 - \omega) D + (\omega - \sigma) T + \omega S) x^k + \omega b, \quad k = 0, 1, \dots,$$

i.e.

$$x^{k+1} = M_{\sigma,\omega} x^k + d,$$

where

$$M_{\sigma,\omega} = ((D - \sigma T))^{-1} ((1 - \omega) D + (\omega - \sigma) T + \omega S).$$

3 . Convergence result

From now on we shall investigate the convergence of the AOR method for the following clases of linear systems:

$$\mathbf{K_f} : |a_{ii}| > f_i(A), \quad i = 1, 2, \ldots n, \tag{6}$$

where $f = [f_1, \ldots, f_n]^T$ is a G-function.

We already know that each $\mathbf{K_f}$ is a subclass of H-matrices, so that the AOR method is convergent if $\omega \in (0, 1]$ and $\sigma \in [0, 1]$. Here we shall improve this convergence area.

Only because of some technical reasons, we shall suppose that $D = E, E$ is the identity matrix.

Let x be an arbitrary positive vector, $< i >$ defined above (related to the matrix A) and

$$l_i = \frac{1}{x_i} \sum_{j<i, j \in <i>} |a_{ij}| x_j, \quad u_i = \frac{1}{x_i} \sum_{j>i, j \in <i>} |a_{ij}| x_j, \quad i = 1, 2, \ldots n.$$

Theorem 2. *Let* $1 - |\sigma| l_i > 0, \quad i = 1, 2, \ldots, n$. *Then*

$$\rho(M_{\sigma,\omega}) \leq \max_{1 \leq i \leq n} \frac{|1 - \omega| + (|\omega - \sigma|) l_i + |\omega| u_i}{1 - |\sigma| l_i}$$

Proof. Let us suppose that there exists an eigenvalue λ of the matrix $M_{\sigma,\omega}$, such that

$$|\lambda| > \frac{|1 - \omega| + (|\omega - \sigma|) l_i + |\omega| u_i}{1 - |\sigma| l_i}, \quad i = 1, 2, \ldots n.$$

Then for each $i = 1, 2, \ldots n$, it follows that

$$|\lambda - 1 + \omega| > |\omega + \sigma(\lambda - 1)| l_i + |\omega| u_i,$$

which means that for the matrix

$$C = (\lambda - 1 + \omega) E - (\omega + \sigma(\lambda - 1)) T - \omega S$$

we have

$$|c_{ii}| > \hat{r}_i^x(C), \quad i = 1, 2, \dots n,$$

because for each $i = 1, 2, \dots n$, the set $< i >$ for the matrix C is the subset of $< i >$ for the matrix A. Since for each positive $x, \hat{r}^x$ is a G-function, the matrix C is a regular matrix.

But, this is a contradiction with the fact that λ is an eigenvalue of the matrix $M_{\sigma,\omega}$, i.e.

$$det(\lambda E - M_{\sigma,\omega}) = det(E - \sigma T)^{-1} detC = 0.$$

The proof is completed. □

Now, we shall return to the clases (6) :

$$\mathbf{K_f} : |a_{ii}| = 1 > f_i(A), \quad i = 1, 2, \dots n,$$

f is a G-function.

Let x be the positive vector from the Theorem 1. Then

$$1 > f_i(A) \geq \hat{r}_i^x(A) = l_i + u_i, \quad i = 1, 2, \dots n,$$

and the following theorem can be easily proved.

Theorem 3. *Let $A \in \mathbf{K_f}$. If l_i and u_i are defined above, then the AOR method is convergent for:*

$(i) 0 < \omega \leq 1, 0 \leq \sigma \leq 1$ *or*

$(ii) 0 < \omega \leq 1, -\min_i \frac{\omega(1-l_i-u_i)}{2l_i} < \sigma < \min_i \frac{\omega(1+l_i-u_i)}{2l_i}$ *or*

$(iii) 1 \leq \omega < \min_i \frac{2}{1+l_i+u_i}, -\min_i \frac{2-\omega(1+l_i+u_i)}{2l_i} < \sigma < \min_i \frac{2-\omega(1-l_i+u_i)}{2l_i}$.

Proof. The statement (i) follows from the convergence result for H-matrices. If the parameters ω and σ satisfy the condition (ii) or (iii), then from Theorem 2 it follows that $\rho(M_{\sigma,\omega}) < 1$.□

4. Numerical example

In order to compare our result with the previous one from [2], we shall consider the following strictly diagonally dominant matrix

$$A = \begin{bmatrix} 1 & -0.25 & -0.25 & 0 \\ -0.25 & 1 & -0.25 & -0.25 \\ 0 & 0 & 1 & -0.25 \\ 0 & 0 & -0.25 & 1 \end{bmatrix}.$$

For the matrix A we have:

$$<1>=<2>=\{1,2\}, <3>=<4>=\{3,4\},$$

and the condition

$$|a_{ii}| = 1 > r_i(A) \geq \hat{r}_i^x(A), \quad i = 1,2,\dots n,$$

is obviously satisfied with $x = [1,1,\dots 1]^T$.

It is interesting to point out that the upper bound for spectral radius of the JOR iteration matrix ($\mathcal{M}_{0,\omega}$) obtained from Theorem 3 is equal to exact value of the spectral radius.

In the following figure the convergence area obtained from Theorem 3 is shown. It contains the convergence area obtained from the upper bound for spectral radius of the AOR iterative matrix which depends on $r_i(L)$ and $r_i(U)$, which is shaded grey.

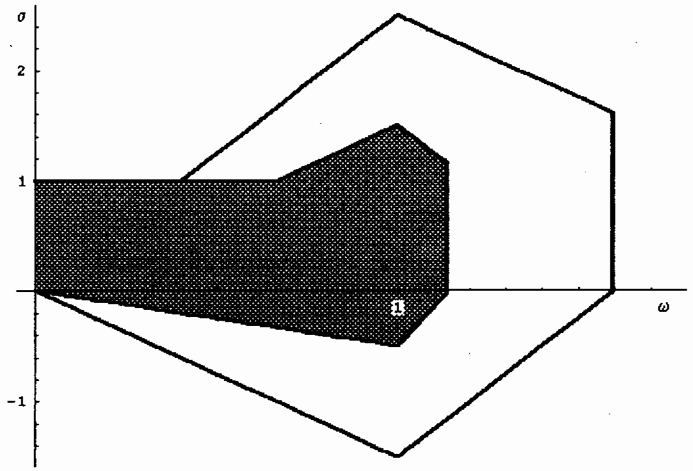

Figure 1

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REZIME

KAKO PRIMENITI TEORIJU G-FUNKCIJA NA ANALIZU KONVERGENCIJE RELAKSACIONIH POSTUPAKA

Pomoću teorije G-funkcija dobijena je oblast konvergencije za AOR (Accelerated Overrelaxation) postupak za rešavanje sistema linearnih jednačina specijalnog oblika. Preciznije, dokazana je nova gornja granica za spektralni radijus AOR iterativne matrice. Rezultat o konvergenciji je generalizacija tvrdjenja iz [7].

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