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LOCAL AND GLOBAL CONVERGENCE OF THE UNSYMMETRIC SOR - NEWTON METHOD

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Abstract

For solving systems of nonlinear equations a combination of the unsymmetric SOR method and Newton's method is proposed. Local and global convergence for some special nonlinear systems is proved.

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1. Introduction

Many problems require the numerical solution of a system of n nonlinear equation with n unknowns, i.e. for a given mapping

$$F : D \subset \mathcal{R}^n \to \mathcal{R}^n,$$

we search $x^* \in \mathcal{R}^n$ such that

$$F(x^*) = 0,$$

or equivalently

$$f_i(x^*) = 0, \quad i = 1, \ldots, n,$$

where $f_i : D \subset \mathcal{R}^n \to \mathcal{R},\ \ i = 1, \ldots, n.$

Such problems are usually solved by using some iterative methods. A combination of the Unsymmetric Successive Overrelaxation method and Newton method is considered in this paper.

Nonlinear Unsymmetric SOR is a two half iteration method which can be expressed in the following form

1. step: Solve the equation

$$f_i(x_1^{k+\frac{1}{2}}, \ldots, x_{i-1}^{k+\frac{1}{2}}, \tilde{x}_i, x_{i+1}^k, \ldots, x_n^k) = 0,$$

2. step: Calculate the first half iteration $x_i^{k+\frac{1}{2}}$

$$x_i^{k+\frac{1}{2}} = \sigma \tilde{x}_i + (1-\sigma) x_i^k$$

1. and 2. step are applied for $i = 1, \ldots, n$.

3. step: Solve the equation

$$f_i(x_1^{k+\frac{1}{2}}, \ldots, x_{i-1}^{k+\frac{1}{2}}, \bar{x}_i, x_{i+1}^{k+1}, \ldots, x_n^{k+1}) = 0,$$

4. step: Calculate the new iteration x_i^{k+1}

$$x_i^{k+1} = \omega \bar{x}_i + (1-\omega) x_i^{k+\frac{1}{2}}$$

3. and 4. step are applied for $i = n, \ldots, 1$.

Here $\omega \neq 0, \sigma \neq 0$ are real parameters.

If we solve nonlinear equations

$$f_i(x_1^{k+\frac{1}{2}}, \ldots, x_{i-1}^{k+\frac{1}{2}}, \tilde{x}_i, x_{i+1}^k, \ldots, x_n^k) = 0,\ i = 1, \ldots, n,$$

$$f_i(x_1^{k+\frac{1}{2}}, \ldots, x_{i-1}^{k+\frac{1}{2}}, \bar{x}_i, x_{i+1}^{k+1}, \ldots, x_n^{k+1}) = 0,\ i = n, \ldots, 1,$$

with one step of Newton's method iteration rule for the Unsymmetric SOR - Newton (USSORN) method is obtained:

(1) $$x_i^{k+\frac{1}{2}} = x_i^k - \sigma \frac{f_i(x^{k,i})}{\frac{\partial f_i}{\partial x_i}(x^{k,i})}, \quad i = 1, \ldots, n$$

$$x^{k,i} = [x_1^{k+\frac{1}{2}}, \ldots, x_{i-1}^{k+\frac{1}{2}}, x_i^k, x_{i+1}^k, \ldots, x_n^k]^T$$

(2) $$x_i^{k+1} = x_i^{k+\frac{1}{2}} - \omega \frac{f_i(\bar{x}^{k,i})}{\frac{\partial f_i}{\partial x_i}(\bar{x}^{k,i})}, \quad i = n, \ldots, 1$$

$$\bar{x}^{k,i} = [x_1^{k+\frac{1}{2}}, \ldots, x_i^{k+\frac{1}{2}}, x_{i+1}^{k+1}, \ldots, x_n^{k+1}]^T.$$

Such combination of the well - known iterative methods for solving linear systems of equations and Newton's method has been considered in many papers, for example SORN method - [11], MSORN - [13], AORN and MAORN methods - [6], [15].

2. Local convergence

Iteration rules (1) and (2) can be written in the following form

$$x^{k+\frac{1}{2}} = \hat{\mathcal{L}}_\sigma(x^k),$$

$$x^{k+1} = \hat{\mathcal{U}}_\omega(x^{k+\frac{1}{2}}).$$

If

$$\hat{\mathcal{S}}_{\sigma\omega} = \hat{\mathcal{U}}_\omega \hat{\mathcal{L}}_\sigma,$$

USSORN iterations are generated by

(3) $$x^{k+1} = \hat{\mathcal{S}}_{\sigma\omega}(x^k), \quad k = 0, 1, \ldots.$$

The system $F(x) = 0$ has the solution x^* iff x^* is the fixed point of the mapping $\hat{\mathcal{S}}_{\sigma\omega}$.

$\hat{\mathcal{L}}_\sigma$ is the operator of SORN method and as it is shown in paper [11] F - derivative of $\hat{\mathcal{L}}_\sigma$ at x^* is

$$\hat{\mathcal{L}}'_\sigma(x^*) = (E - \sigma L(x^*))^{-1}((1-\sigma)E + \sigma U(x^*)).$$

Here, E denotes identity matrix, and

$$F'(x^*) = D(x^*)(E - L(x^*) - U(x^*))$$

is the standard splitting of the matrix $F'(x^*)$ into diagonal matrix $D(x^*)$, strictly lower triangular matrix $L(x^*)$ and strictly upper triangular matrix $U(x^*)$.

As $\hat{\mathcal{U}}_\omega$ is the backward SORN operator, it is easy to prove that

$$\hat{\mathcal{U}}'_\omega(x^*) = (E - \omega U(x^*))^{-1}((1-\omega)E + \omega L(x^*)).$$

Now, we can conclude that the operator $\hat{\mathcal{S}}_{\sigma\omega}$ is F - differentiable at x^* and

$$\hat{\mathcal{S}}'_{\sigma\omega}(x^*) = \hat{\mathcal{U}}'_\omega(\mathcal{L}_\sigma(x^*))\hat{\mathcal{L}}'_\sigma(x^*) = \hat{\mathcal{U}}'_\omega(x^*)\hat{\mathcal{L}}'_\sigma(x^*),$$

i.e.

$$\begin{aligned}\hat{\mathcal{S}}'_{\sigma\omega}(x^*) = {} & (E - \omega U(x^*))^{-1}((1-\omega)E + \omega L(x^*)\cdot \\ & \cdot(E - \sigma L(x^*))^{-1}((1-\sigma)E - \sigma U(x^*)).\end{aligned}$$

Obviously, $\hat{\mathcal{S}}'_{\sigma\omega}(x^*)$ has the form of the USSOR iteration matrix for the linear system with the matrix $F'(x^*)$.

By the Ostrowski's theorem ([21]) sufficient condition for local convergence of the USSORN method is

$$\rho(\hat{\mathcal{S}}'_{\sigma\omega}(x^*)) < 1.$$

So, all sufficient conditions for convergence of the USSOR method in linear case can be generalized to the nonlinear case as sufficient conditions for the local convergence of the USSORN method. Next theorem is an example of such generalization.

Theorem 1. *Let F be an F - differentiable mapping at x^*, $F(x^*) = 0$. If $F'(x^*)$ is a strictly diagonally dominant matrix, $\varepsilon = ||L(x^*) + U(x^*)||_\infty$ and $(\sigma, \omega) \in \mathcal{O}(\varepsilon)$, then x^* is the point of attraction for the iterations (3). Area $\mathcal{O}(\varepsilon)$ is defined as*

$$\sigma \in (-\frac{1-\varepsilon}{2\varepsilon}, \frac{\varepsilon+1}{2\varepsilon}),$$

$$\omega \in (max\{\frac{|1-\sigma|+|\sigma|\varepsilon-1}{|1-\sigma|+\varepsilon(|\sigma|+1)}, \frac{|1-\sigma|+|\sigma|\varepsilon-1}{|1-\sigma|(1-\varepsilon)}\}, min\{\frac{1+|1-\sigma|+\varepsilon|\sigma|}{\varepsilon(1+|\sigma|)+|1-\sigma|}, \frac{1+|1-\sigma|-\varepsilon|\sigma|}{|1-\sigma|(1+\varepsilon)}\}) \ .$$

3. Global convergence

From now on, we are going to consider a system of the special form

$$(4) \qquad f_i(x) := \sum_{i=1}^{n} a_{ij}x_j + s_i(x_i) = 0 \ \ i = 1, \ldots, n,$$

where $a_{ij} \in \mathcal{R}$, $i, j = 1, \ldots, n$, $a_{ii} \neq 0$, $i = 1, \ldots, n$ and $s_i(t)$, $i = 1, \ldots, n$, are real nonlinear differentiable functions.

In order to avoid calculation of partial derivatives in (1) and (2), slight modification is proposed:

$$(5) \qquad x_i^{k+\frac{1}{2}} = x_i^k - \sigma\frac{f_i(x^{k,i})}{a_{ii}}, \ \ i = 1, \ldots, n$$

$$(6) \qquad x_i^{k+1} = x_i^{k+\frac{1}{2}} - \omega\frac{f_i(\bar{x}^{k,i})}{a_{ii}}, \ \ i = n, \ldots, 1,$$

If $\tilde{\mathcal{L}}_\sigma$ denotes operator for (5), and $\tilde{\mathcal{U}}_\omega$ operator for (6), then the modified USSORN method can be written as

$$(7) \qquad x^{k+1} = \tilde{\mathcal{S}}_{\sigma\omega}(x^k) = \tilde{\mathcal{U}}_\omega(\tilde{\mathcal{L}}_\sigma(x^k)), \ \ k = 0, 1, \ldots .$$

We shall use the following notation.

$$p_{1i}(\alpha) = \sum_{j=1}^{i-1} |\frac{a_{ij}}{a_{ii}}|(|1-\alpha| + |\alpha| p_{1j}(\alpha)) + \sum_{j=i+1}^{n} |\frac{a_{ij}}{a_{ii}}|, \ \ i = 1, \ldots, n,$$

$$p_{2i}(\alpha) = |\alpha| \sum_{j=1}^{i-1} |\frac{a_{ij}}{a_{ii}}| p_{2j}(\alpha) + 1, \ \ i = 1, \ldots, n,$$

$$q_{1i}(\alpha) = \sum_{j=1}^{i-1} |\frac{a_{ij}}{a_{ii}}| + \sum_{j=i+1}^{n} |\frac{a_{ij}}{a_{ii}}|(|1-\alpha| + |\alpha| q_{1j}(\alpha)), \;\; i = n, \ldots, 1,$$

$$q_{2i}(\alpha) = |\alpha| \sum_{j=i+1}^{n} |\frac{a_{ij}}{a_{ii}}| q_{2j}(\alpha) + 1, \;\; i = n, \ldots, 1.$$

Theorem 2. *Let the following conditions be satisfied:*

(a) $|s_i'(t)| \leq \gamma, \;\; t \in \mathcal{R}, \;\; i = 1, \ldots, n,$

(b) $|a_{ii}| \geq a > 0, \;\; i = 1, \ldots, n,$

(c) $\delta = \delta_1 \delta_2 < 1$, *with*

$$\delta_1 = \max_{1 \leq i \leq n} \{|1-\sigma| + |\sigma| p_{1i}(\sigma) + \frac{|\sigma|}{a} \gamma p_{2i}(\sigma)\},$$

$$\delta_2 = \max_{1 \leq i \leq n} \{|1-\omega| + |\omega| q_{1i}(\omega) + \frac{|\omega|}{a} \gamma q_{2i}(\omega)\}.$$

Then, for every $x^0 \in \mathcal{R}^n$, *the sequence* $\{x^k\}$ *generated by (7) converges to the solution* x^* *of system (4).*

Proof. By means of mathematical induction it can be proved that the following inequalities

$$|(\tilde{\mathcal{U}}_\omega(x))_i - (\tilde{\mathcal{U}}_\omega(y))_i| \leq \delta_2 ||x-y||_\infty, \;\; i = 1, \ldots, n,$$

$$|(\tilde{\mathcal{L}}_\sigma(x))_i - (\tilde{\mathcal{L}}_\sigma(y))_i| \leq \delta_1 ||x-y||_\infty, \;\; i = 1, \ldots, n,$$

are valid for every $x, y \in \mathcal{R}^n$. So,

$$||\tilde{\mathcal{U}}_\omega(x) - \tilde{\mathcal{U}}_\omega(y)||_\infty \leq \delta_2 ||x-y||_\infty,$$

$$||\tilde{\mathcal{L}}_\sigma(x) - \tilde{\mathcal{L}}_\sigma(y)||_\infty \leq \delta_1 ||x-y||_\infty,$$

and

$$||\tilde{\mathcal{S}}_{\sigma\omega}(x) - \tilde{\mathcal{S}}_{\sigma\omega}(y)||_\infty \leq \delta_2 ||\tilde{\mathcal{L}}_\sigma(x) - \tilde{\mathcal{L}}_\sigma(y)||_\infty \leq \delta_2 \delta_1 ||x-y||_\infty.$$

As $\delta_1\delta_2 < 1$, the mapping $\tilde{S}_{\sigma\omega}$ is a contraction, which completes the proof. □

Convergence area can be derived in the following way. Let

$$p(\sigma) = \max_{1\le i\le n}\{p_{1i}(\sigma) + (\gamma/a)p_{2i}(\sigma)\},$$

$$q(\omega) = \max_{1\le i\le n}\{q_{1i}(\omega) + (\gamma/a)q_{2i}(\omega)\},$$

and

$$l_i = \sum_{j=1}^{i-1}|\frac{a_{ij}}{a_{ii}}|, \quad u_i = \sum_{j=i+1}^{n}|\frac{a_{ij}}{a_{ii}}|, \quad i = 1,\dots,n.$$

After some calculations, we obtain

$$p(\sigma) \le |1-\sigma|l_i + u_i + (\gamma/a) + |\sigma|l_i p(\sigma),$$

i.e.

$$p(\sigma) \le \max_{1\le i\le n}\frac{|1-\sigma|l_i + u_i + \gamma/a}{1-|\sigma|l_i}, \tag{8}$$

and

$$q(\omega) \le \max_{1\le i\le n}\frac{|1-\omega|u_i + l_i + \gamma/a}{1-|\omega|u_i}. \tag{9}$$

Theorem 3. *Let*

(a) $|s_i'(t)| \le \gamma, \; t\in\mathcal{R}, \; i = 1,\dots,n,$

(b) $|a_{ii}| \ge a > 0, \; i = 1,\dots,n.$

and parameters σ *and* ω *be chosen such that*

(c) $1-|\sigma|l_i > 0, \; 1-|\omega|u_i > 0, \; i = 1,\dots,n$ *and*

$$\delta^* = \max_{1\le i,j\le n}\frac{|1-\sigma| + |\sigma|(u_i+\gamma/a)}{1-|\sigma|l_i}\cdot\frac{|1-\omega| + |\omega|(l_j+\gamma/a)|}{1-|\omega|u_j} < 1.$$

Then the sequence $\{x^k\}$ *generated by the iteration rule (7) converges to the solution* x^* *of the system (4) for every* $x^0 \in \mathcal{R}^n$.

Proof. This theorem is an immediate corollary of Theorem 2 and inequalities (8) and (9).

If $\gamma = 0$, the system (4) becomes linear, and the statement of Theorem 3 is given in the paper [9].

4. Numerical example

Global convergence area for the modified unsymmetric SORN method obtained in Theorem 3 is given at figure 1. The system of equations

$$Ax + S(x) = 0,$$

where $S(x) = [s_1(x), \dots, s_n(x)]^T$ is a diagonal mapping , i.e. $s_i(x) = s_i(x_i)$, and $|s_i'(t)| \leq \gamma,\ \ t \in \mathcal{R}, i = 1, \dots, n$ is considered. The matrix A is block tridiagonal matrix:

$$A = \begin{bmatrix} R & -E & 0 & \dots & 0 \\ -E & R & -E & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & R & -E \\ 0 & 0 & \dots & -E & R \end{bmatrix},$$

with the matrix R

$$R = \begin{bmatrix} 8 & -1 & 0 & \dots & 0 \\ -1 & 8 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 8 & -1 \\ 0 & 0 & \dots & -1 & 8 \end{bmatrix}.$$

Figure 1 shows the convergence area for unsymmetric SORN method for different values of the parameter γ.

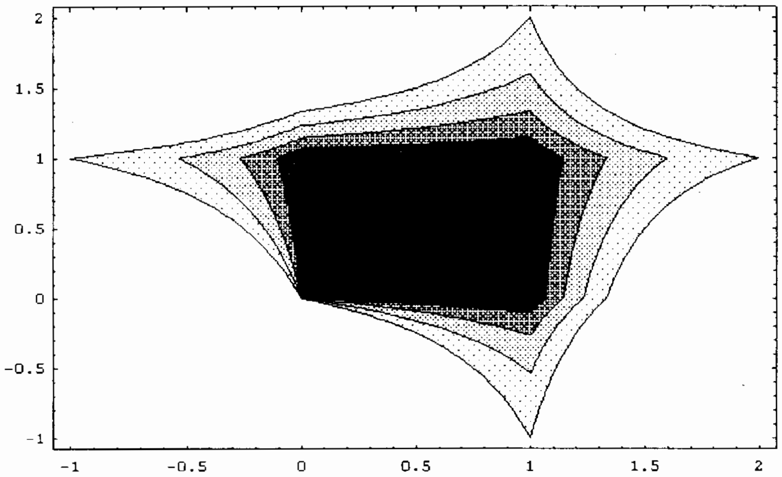
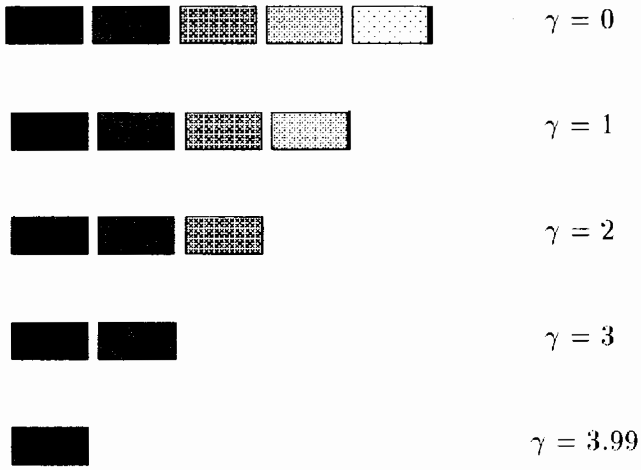


Figure 1



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REZIME

LOKALNA I GLOBALNA KONVERGENCIJA NESIMETRIČNOG SOR - NJUTNOVOG POSTUPKA

Za rešavanje sistema nelinearnih jednačina koristi se kombinacija nesimetričnog SOR postupka sa Njutnovim postupkom i jedna njena modifikacija. Dokazuje se lokalna i globalna konvergencija za sisteme nelinearnih jednačina specijalnog obl

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