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ON NUMERICAL SOLUTION OF SEMILINEAR SINGULAR PERTURBATION PROBLEMS BY USING THE HERMITE SCHEME

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Abstract

The semilinear singularly perturbed boundary value problem is solved numerically by a finite-difference method which uses a combination of the Hermite scheme and the standard central scheme on a special non-equidistant mesh. The method is a modification of that given in [6]. We prove the same result (fourht order accuracy uniform in the perturbation parameter), but without the constraint on the non-linearity which was used in [6].

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1. Introduction

We shall consider a numerical method for the following singularly perturbed boundary value problem:

(1.1a) $$Tu := -\varepsilon^2 u'' + c(x,u) = 0, \quad x \in I = [0,1],$$

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$$(1.1b) \qquad Bu := (u(0), u(1)) = (0,0),$$

where $\varepsilon \in (0, \varepsilon^*]$, usually $\varepsilon^* << 1$. This problem arises in practice, see [2], [3]. It was treated numerically in various papers (let us mention [4-8] and for other references see [6] where a survey was given) under appropriate smoothness assumptions on c and the standard condition:

$$(1.2) \qquad c_u(x,u) \geq c_* > 0, \quad x \in I, \quad u \in \mathbf{R}.$$

Under this condition the solution to (1.1) has in general two boundary layers of width $O(\varepsilon)$. Because of that we shall use a non-equidistant discretization mesh which is dense in the layers. A modification of the finite-difference method from [6] will be applied.

The method from [6] uses the fourth order Hermite scheme on a special non-equidistant mesh, which was introduced in [7]. The mesh is generated by a rational function which maps equidistant points into appropriate mesh points. Such an approach was introduced in [1], but a more complicated logarithmic mesh generating function was applied there. In [6], because of the stability reasons, the Hermite scheme is abandoned at some mesh points, and the standard second order central scheme is used. Nevertheless, the combination has the fourth order uniform accuracy (troughout the paper by *uniform* we shall mean *uniform in* ε). Essentially, this is because the central scheme is used outside the layers, where $\varepsilon^2|u''|$ is small. The same method was used in [5], where its accuracy was improved to the sixth order by the Richardson extrapolation. The Hermite scheme was also used in [4] in combination with the solution to the reduced problem corresponding to (1.1).

In addition to (1.2), the method from [6] requires the following conditions:

$$(1.3a) \qquad c_u(x,u) \leq c^*, \quad x \in I, \quad u \in \mathbf{R},$$

$$(1.3b) \qquad 5c_* - 2c^* > 0,$$

(note that in fact the condition on c_u was formulated in [6] in a different way – however, (1.3) describes its essence). Obviously, the condition (1.3) is an unpleasant restriction on the nonlinearity of the function c. It was introduced in [6] in order to prove the uniform stability of the discrete problem. Our aim here will be to modify the method from [6] in such a way that (1.3b) will not be needed. In addition to that, the smoothness assumption on c will be relaxed, as well as the conditions (1.2) and (1.3a). Precise assumptions will be given in the next section.

Let us briefly explain the way in which we shall avoid (1.3b). Of course, we have to know where that condition was used in [6]. The Fréchet derivative of the discrete operator from [6] is not an L−matrix, but Theorem 1 from [6] shows that it is strictly diagonally dominant. The condition (1.3b) is used in that analysis. We can see from the proof that (1.3b) is not needed in one case (case II.1), and that is the case when we have the L−form. This gives us the idea to insist on the L−form and to replace the Hermite scheme by the central scheme whenever the Hermit scheme would spoil that form. In this way we shall obtain a discretization which uses the central scheme more than the discretization from [6], but its stability can be proved very easily. On the other hand, the fourth order uniform consistency is more complicated to prove, but we shall do that due to the fact that the central scheme will be used still sufficiently far from the layers. However, for this we shall need a mesh generating function which is smoother that the one from [6]. Our method will be given in Section 3, and in Section 4 we shall give some numerical results.

2. Preliminaries

Throughout the paper we shall assume the following two hypotheses on the problem (1.1), cf. [3, Chapter 3]:

H1. Let the reduced problem $c(x,u)=0,\ \ x\in I$, have a $C^2(I)$−solution u_0.

Then there exist $C^2(I)$−functions d_1 and d_2, independent of ε and such that

$$d_i(x)\geq d_*>0,\ \ i=1,2,\ \ x\in I,$$

$$d_1(t)\geq -u_0(t),\ \ d_2(t)\geq u_0(t),\ \ t=0,1,$$

and we assume:

H2. $c\in C^4(I^\times W),\ \ W=\{(x,u):x\in I,\ \ y(x)\leq u\leq z(x)\},$
$y(x):=u_0(x)-d_2(x),\ \ z(x):=u_0(x)+d_1(x);$
$c^*\geq c_u(x,u)\geq c_*>0,\ \ (x,u)\in W.$

Lemma 1. *There exist a sufficiently small* ε^* *such that for* $\varepsilon \in (0, \varepsilon^*]$ *the problem (1.1) has a solution* u_ε *which satisfies* $(x, u_\varepsilon(x)) \in W$ *and* $u_\varepsilon \in C^6(I)$. *Such a solution is unique.*

Proof. We have

$$z(t) \geq 0, \;\; t = 0, 1,$$

and if ε^* is sufficiently small, it follows:

$$Tz(x) \geq -\varepsilon^2 z''(x) + c_* d_* \geq 0, \;\; x \in I.$$

Thus, z is an upper solution to the problem (1.1). It can be shown similarly that y is a lower solution. This means that u_ε exists and

$$y(x) \leq u_\varepsilon(x) \leq z(x), \;\; x \in I.$$

Such a u_ε is unique because the operator (T,B) is inverse monotone. □

Remark 1. If there exist constants u_* and u^* such that

$$u^* \geq 0 \geq u_*, \;\; c(x, u^*) \geq 0 \geq c(x, u_*), \;\; x \in I,$$

then we can take $y \equiv u_*$, $z \equiv u^*$. Then in Lemma 1 we do not need the assumption that ε^* is sufficiently small. For such a reasoning cf. [10].

Throughout the paper we shall denote by M any (in the sense of O(1)) positive constant which is independent of ε. Later on, these constants will be independent of the discretization mesh as well. Let $0 < \gamma < c_*^{\frac{1}{2}}$. Then we have:

Lemma 2. *The following estimates hold for* $x \in I$:

(2.1a) $$|u_\varepsilon^{(k)}(x)| \leq M\{1 + \varepsilon^{-k}[\exp(-\gamma x/\varepsilon) + \exp(\gamma(x-1)/\varepsilon)]\},$$

(2.1b) $$k = 0(1)4,$$

(2.1c) $$|u_\varepsilon^{(k)}(x)| \leq M\{\varepsilon^{4-k} + \varepsilon^{-k}[\exp(-\gamma x/\varepsilon) + \exp(\gamma(x-1)/\varepsilon)]\},$$

(2.1d) $$k = 5, 6.$$

Proof. The estimates (2.1a) follow from [7]. Then differentiate (1.1a) four times to get the estimate for $k = 6$. To estimate $|u_\varepsilon^{(5)}(x)|$ we shall use a technique from [1]. Let $g \in C^2(I)$. Then for $\delta_i \in I$, $i = 1, 2$, $\delta_1 < \delta_2$, it holds that

$$|g'(t)| \leq \frac{|g(\delta_1)| + |g(\delta_2)|}{\delta_2 - \delta_1} + (\delta_2 - \delta_1) \max_{\delta_1 \leq s \leq \delta_2} |g''(s)|, \quad t \in [\delta_1, \delta_2].$$

This follows from the expansion

$$g(\delta_2) - g(\delta_1) = (\delta_2 - \delta_1) g'(t) + \frac{(\delta_2 - t)^2}{2} g''(\Theta_2) - $$
$$- \frac{(\delta_1 - t)^2}{2} g''(\Theta_1), \quad \delta_1 < \Theta_1 < t, \quad t < \Theta_2 < \delta_2.$$

Next, set $g = u_\varepsilon^{(4)}$. If $x \in [0, \frac{1}{2}]$ take $\delta_1 = x$, $\delta_2 = x + \frac{\varepsilon}{2\varepsilon^*} \leq 1$, and if $x \in [\frac{1}{2}, 1]: \; \delta_1 = x - \frac{\varepsilon}{2\varepsilon^*} \geq 0$, $\delta_2 = x$. Then use the estimates of $|u_\varepsilon^{(4)}(x)|$ and $|u_\varepsilon^{(6)}(x)|$ to get (2.1b) for $k = 5$. □

Note that $c \in C^4(I \times \mathbf{R})$ was used in [7] in order to prove the second order uniform accuracy, while [6] used a stronger smoothness assumption for the fourth order accuracy (in fact, instead of (2.1b) estimates of type (2.1a) were used in [6] for $k = 5, 6$ as well, but for that $c \in C^6(I \times \mathbf{R})$ was needed). Also note that in H2 the conditions (1.2) and (1.3a) are relaxed, while (1.3b) has not been assumed at all.

Let us turn to the discretization mesh. The mesh points will be given by:

(2.2)
$$x_i = \lambda(t_i), \quad t_i = ih, \quad i = 0(1)n,$$
$$h = \frac{1}{n}, \quad n = 2m, \quad m \in \mathbf{N} \setminus \{1\},$$

where:

$$\lambda(t) = \begin{cases} \mu(t) := \frac{a\varepsilon t}{q - t}, & t \in [0, \alpha] \\ \pi(t) := \omega(t - \alpha)^3 + \frac{\mu''(\alpha)(t-\alpha)^2}{2} + \mu'(\alpha)(t - \alpha) + \mu(\alpha), & t \in [\alpha, \frac{1}{2}] \\ 1 - \lambda(1 - t), & t \in [\frac{1}{2}, 1] \end{cases}$$

Here q is an arbitrary parameter from $((\varepsilon^*)^{\frac{1}{3}}, \frac{1}{2})$ and $\alpha = q - \varepsilon^{\frac{1}{3}} > 0$, where we assume that $\varepsilon^* < \frac{1}{8}$. The coefficient ω is determined from $\pi(\frac{1}{2}) = \frac{1}{2}$:

$$\omega = (\frac{1}{2} - \alpha)^{-3} \{ \frac{1}{2} - a[q(\frac{1}{2} - \alpha)^2 + q(\frac{1}{2} - \alpha)\varepsilon^{\frac{1}{3}} + \alpha \varepsilon^{\frac{2}{3}}]\},$$

and a is chosen so that $\omega \geq 0$ (such an a, independent of ε, obviously exists).

The function λ from [6] uses μ and a tangent line for π, so that $\lambda \in C^1(I)$ and λ'' is discontinuous and unbounded when $\varepsilon \to 0$. By our choice of α and π we get $\lambda \in C^2(I \backslash \{\frac{1}{2}\})$ and:

(2.3a) $$|\lambda'(t)| \leq M, \quad t \in I,$$

(2.3b) $$|\lambda''(t)| \leq M, \quad t \in I \backslash \{\frac{1}{2}\},$$

which we shall need in our analysis. A similar function λ was used in [9], and in [5] even a smoother function was required.

Let

$$h_i = x_i - x_{i-1}, \quad i = 1(1)n,$$

$$\overline{h_i} = \frac{h_i + h_{i+1}}{2}, \quad i = 1(1)n-1.$$

By w^h, v^h etc. we shall denote mesh functions on the mesh (2.2). They will be identified with $\mathbf{R}^{n+1}$–column-vectors:

$$w^h = [w_0, w_1, \ldots w_n]^T, \quad (w_i := w_i^h).$$

In particular, we shall take:

$$e^h = [1, 1, \ldots 1]^T.$$

Let g be an arbitrary $C(I)$–function. Then:

$$g^h := [g_0, g_1, \ldots g_n]^T, \quad g_i := g(x_i), \quad i = 0(1)n.$$

Thus we shall have $u_\varepsilon^h, y^h, z^h$. The numerical approximation to u_ε^h will be denoted by w_ε^h.

By $\|\cdot\|$ we shall denote the standard maximum vector norm:

$$\|w^h\| = \max_{0 \leq i \leq n} |w_i|,$$

and the corresponding matrix norm. Let

$$w^h = \{w^h : y^h \leq w^h \leq z^h\},$$

where the inequality sign in $\mathbf{R}^{n+1}$ should be understood componentwise.

Finally, let us introduce some finite-difference schemes:

- the scheme for approximation of the second derivative:
  $Dw_i = [(w_{i+1} - w_i)/h_{i+1} + (w_{i-1} - w_i)/h_i]/\overline{h_i}$;

- the central scheme corresponding to the operator T :
  $T_C w_i = -\varepsilon^2 Dw_i + c(x_i, w_i)$;

- the Hermite scheme corresponding to the operator T, see [4-6]:
  $T_H w_i = -\varepsilon^2 Dw_i + b_i^- c_{i-1} + b_i c_i + b_i^+ c_{i+1}$,

where

$$c_j := c(x_j, w_j),$$

$$b_i^- = \frac{h_i^2 - h_{i+1}^2 + h_i h_{i+1}}{12 h_i \overline{h_i}},$$

$$b_i^+ = \frac{h_{i+1}^2 - h_i^2 + h_i h_{i+1}}{12 h_{i+1} \overline{h_i}},$$

$$b_i = 1 - b_i^- - b_i^+ = \frac{h_i^2 + h_{i+1}^2 + 3 h_i h_{i+1}}{6 h_i h_{i+1}}.$$

3. The Numerical Method

Let us introduce the discretization of the problem (1.1) on the mesh (2.2):

(3.1a) $$F w^h = 0,$$

(3.1b) where

(3.1c) $$F_0 w^h = w_0,$$

(3.1d) $$F_i w^h = \begin{cases} T_H w_i & \text{if } b_i^- \geq 0,\ b_i^+ \geq 0 \text{ and } \rho_i \leq 1, \\ T_C w_i & \text{otherwise,} \end{cases}$$

(3.1e) $$i = 1(1)n-1$$

(3.1f) $$F_n w^h = w_n.$$

Here:

$$\rho_i = \frac{[(h_{i+1} + h_i)|h_{i+1} - h_i| + h_i h_{i+1}]c^*}{12\varepsilon^2}.$$

Theorem 1. *Let ε^* be sufficiently small. Then for $\varepsilon \in (0, \varepsilon^*]$ the discrete problem (3.1) has a unique solution w_ε^h in W^h. Moreover, the following stability inequality holds for any w^h and v^h from W^h :*

$$\|w^h - v^h\| \le c_*^{-1} \|Fw^h - Fv^h\|. \tag{3.2}$$

Proof. For the technique cf. [10]. The important thing is that the switching between T_H and T_C does not depend on w^h. Thus, the Fréchet derivative $A := F'(w^h)$, $w^h \in W^h$, is well defined. Let $A = [a_{ij}]$. The non-zero elements of this tridiagonal matrix are:

$$a_{00} = 1, \quad a_{nn} = 1,$$

and for $i = 1(1)n-1$:

$$a_{ii} = \frac{2\varepsilon^2}{h_i h_{i+1}} + p_i c_{u,i},$$

$$a_{i,i-1} = -\frac{\varepsilon^2}{h_i \bar{h}_i} + p_i^- c_{u,i-1},$$

$$a_{i,i+1} = -\frac{\varepsilon^2}{h_{i+1} \bar{h}_i} + p_i^+ c_{u,i+1},$$

where

$$c_{u,j} = c_u(x_j, w_j),$$

$$p_i^\pm = \begin{cases} b_i^\pm & \text{if } T_H \text{ is applied at } x_i, \\ 0 & \text{if } T_C \text{ is applied at } x_i, \end{cases}$$

$$p_i = 1 - p_i^- - p_i^+.$$

It is obvious that we have

$$a_{ii} > 0, \quad i = 0(1)n,$$

and whenever T_C is used it holds that

$$a_{i,i\pm 1} \le 0.$$

However, the last inequality holds also when T_H is used. Indeed, because of

$$b_i^- \ge 0 \text{ and } \rho_i \le 1$$

we have

$$a_{i,i-1} \leq -\frac{\varepsilon^2}{h_i \overline{h_i}} + b_i^- c^* \leq \frac{\varepsilon(-1+\rho_i)}{h_i \overline{h_i}} \leq 0,$$

and

$$a_{i,i+1} \leq 0$$

holds in a similar way. Thus A is an L-matrix. Moreover, A is an M-matrix since we have

$$Ae^h \geq c_* e^h, \tag{3.3}$$

(note that for this it is again important that p_i^-, $p_i^+ \geq 0$).

Now the existence of w_ε^h follows if we show:

$$Fz^h \geq 0 \geq Fy^h.$$

Let us prove the first inequality (the second one can be handled similarly). It is obvious that

$$F_0 z^h,\ F_n z^h \geq 0,$$

and for

$$F_i z^h \geq 0,\ i = 1(1)n-1,$$

a sufficiently small ε^* is needed, cf. the proof of Lemma 1. Indeed, for some $\omega_i \in (x_{i-1}, x_{i+1})$ we have:

$$\begin{aligned} F_i z^h &= -\varepsilon^2 z''(\omega_i) + p_i^- c(x_{i-1}, z_{i-1}) + p_i c(x_i, z_i) + \\ &+ p_i^+ c(x_{i+1}, z_{i+1}) \geq -\varepsilon^2 z''(\omega_i) + (p_i^- + p_i + p_i^+) c_* d_* = \\ &= -\varepsilon^2 z''(\omega_i) + c_* d_* \geq 0. \end{aligned}$$

Finally, let us prove (3.2). From (3.3) it follows that

$$\|A^{-1}\| \leq \frac{1}{c_*}.$$

The same estimate holds for a matrix P of the form:

$$P = \int_0^1 F'(v^h + s(w^h - v^h))ds,$$

for any w^h, $v^h \in W^h$. Then (3.2) follows from

$$w^h - v^h = P^{-1}(w^h - v^h). \qquad \square$$

We can now formulate the fourth order uniform convergence result.

Theorem 2. *Let ε^* be sufficiently small. Then for $\varepsilon \in (0, \varepsilon^*]$ it holds that*

$$\|w_\varepsilon^h - u_\varepsilon^h\| \le Mh^4.$$

Proof. Because of (3.2) it is sufficient to prove

$$\|r^h\| \le Mh^4,$$

where

$$r^h = Fu_\varepsilon^h.$$

First we shall prove

$$|r_i| \le Mh^4, \tag{3.4}$$

for $i = 1(1)m-1$. The same technique as in [1], [4-9] will be used. The proof will be divided into several steps.

Let $i = 1(1)m-1$, thus $[x_{i-1}, x_{i+1}] \subset [0, \frac{1}{2}]$. Because of that estimates (2.1) will be used for $x \in [0, \frac{1}{2}]$ only, and in this case they reduce to

$$|u_\varepsilon^{(k)}(x)| \le M(1 + \varepsilon^{-k} v_\varepsilon(x)), \quad k = 0(1)4, \tag{3.5a}$$

$$|u_\varepsilon^{(k)}(x)| \le M(\varepsilon^{k-4} + \varepsilon^{-k} v_\varepsilon(x)), \quad k = 5, 6, \tag{3.5b}$$

$$v_\varepsilon(x) := \exp(-\gamma x/\varepsilon). \tag{3.5c}$$

The first case is:

I. T_H is applied at x_i.

In this case (3.4) can be proved in the same way as in [6], but since the estimates (3.5b) are rougher than in [6] and our function λ is somewhat different, some details will be given. It holds that, see [6]:

$$r_i = \varepsilon^2 (Q_i + R_i + S_i),$$

where

$$Q_i = \frac{(h_{i+1} - h_i)(2h_i^2 + 2h_{i+1}^2 + 5h_i h_{i+1}) u_\varepsilon^{(5)}(x_i)}{180},$$

$$R_i = -\frac{(h_i^5 + h_{i+1}^5) u_\varepsilon^{(6)}(\alpha_i)}{360(h_i + h_{i+1})},$$

$$S_i = \frac{(h_i^4 + h_{i+1}^4 - h_i^2 h_{i+1}^2) u_\varepsilon^{(6)}(\beta_i)}{144},$$

$$\alpha_i,\ \beta_i \in (x_{i-1}, x_{i+1}).$$

From (2.3a) it follows that

$$h_i \le h_{i+1} \le h\lambda'(t_{i+1}) \le Mh. \tag{3.6}$$

Using this and (3.5b), we get:

$$\varepsilon^2|Q_i| \le Mh^2(h_{i+1} - h_i)\lambda'(t_{i+1})^2[\varepsilon + \varepsilon^{-3}v_\varepsilon(x_{i-1})],$$

$$\varepsilon^2|R_i + S_i| \le Mh^4\lambda'(t_{i+1})^4[1 + \varepsilon^{-4}v_\varepsilon(x_{i-1})].$$

I.1 Let $t_{i-1} \ge \alpha$. Then $h_i = h_{i+1}$ and $Q_i = 0$. Moreover, from (3.6) and

$$v_\varepsilon(x_{i-1}) \le v_\varepsilon(\mu(\alpha)) \le \exp(-M/\varepsilon^{1/2}),$$

it follows:

$$\varepsilon^2|R_i + S_i| \le Mh^4,$$

thus (3.4) is proved in this case.

I.2 Let $t_{i-1} < \alpha$ and $t_{i-1} \le q - 3h$. Now use (2.3b) to get

$$h_{i+1} - h_i \le h^2\lambda''(t_{i+1}) \le Mh^2. \tag{3.7}$$

From here and (3.6) it follows:

$$\varepsilon^2|Q_i| \le Mh^4[\varepsilon + \lambda''(t_{i+1})\lambda'(t_{i+1})^2\varepsilon^{-3}v_\varepsilon(x_{i-1})]. \tag{3.8}$$

Then because of $t_{i-1} \le q - 3h$ we have $t_{i+1} < q$. Now for $k = 1, 2$ it holds that

$$\lambda^{(k)}(t) \le \mu^{(k)}(t), \ \ t \in [\alpha, q), \tag{3.9}$$

provided ε^* be sufficiently small, so that

$$(\varepsilon^*)^{\frac{1}{3}}\omega \le aq. \tag{3.10}$$

Indeed, (3.10) guarantees that

$$(\mu - \pi)^{(3)}(t) \ge \mu^{(3)}(\alpha) - 6\omega \ge 0, \ \ t \in [\alpha, q),$$

and it follows:

$$(\mu - \pi)^{(k)}(t) \ge (\mu - \pi)^{(k)}(\alpha) = 0, \ \ t \in [\alpha, q),$$

first for $k = 2$ and then for $k = 1$. Using (3.9) and

$$q - t_{i+1} \geq \frac{q - t_{i-1}}{3},$$

$$v_\varepsilon(x_{i-1}) = \exp(-\gamma a t_{i-1}/(q - t_{i-1})) \leq M \exp(-\gamma a q/(q - t_{i-1})),$$

from (3.8) we obtain:

$$\varepsilon^2 |Q_i| \leq M h^4 [1 + \varepsilon^3 (q - t_{i+1})^{-7} \varepsilon^{-3} v_\varepsilon(x_{i-1})] \leq$$

$$\leq M h^4 [1 + (q - t_{i-1})^{-7} \exp(-\gamma a q/(q - t_{i-1}))] \leq M h^4.$$

Similarly we can show:

$$\varepsilon^2 |R_i + S_i| \leq M h^4$$

and (3.4) follows again.

I.3 The remaining case is $q - 3h < t_{i-1} < \alpha$. Now use

$$|r_i| = \varepsilon^2 | - u''_\varepsilon(\gamma_i) + b_i^- u''_\varepsilon(x_{i-1}) + b_i u''_\varepsilon(x_i) + b_i^+ u''_\varepsilon(x_{i+1})|,$$

$$\gamma_i \in (x_{i-1}, x_{i+1}),$$

to get

$$|r_i| \leq \varepsilon^2 \max_{x_{i-1} \leq x \leq x_{i+1}} |u''_\varepsilon(x)| [1 + b_i^- + b_i + b_i^+] \leq M[\varepsilon^2 + v_\varepsilon(x_{i-1})].$$

Noting that this case is possible only if

$$\varepsilon^{\frac{1}{3}} < 3h,$$

and using

$$v_\varepsilon(x_{i-1}) \leq v_\varepsilon(\mu(q - 3h)) \leq M \exp(-\gamma a q/3h),$$

we get (3.4) in this case too.

II. T_C is applied at x_i.

This is possible if $\rho_i > 1$ or $b_i^- < 0$ (note that $b_i^+ \geq 0$ for $i = 1(1)m - 1$). We shall use:

$$|r_i| \leq M \varepsilon^2 [(h_{i+1} - h_i) |u_\varepsilon^{(3)}(x_i)| + h_{i+1}^2 |u_\varepsilon^{(4)}(\sigma_i)|], \quad (3.11)$$

$$\sigma_i \in (x_{i-1}, x_{i+1}).$$

IIa. Let $\rho_i > 1$.

This means that

$$\varepsilon^2 \le Mh_{i+1}^2 \le Mh^2,$$

and form (3.5a), (3.6) and (3.7) it follows:

$$|r_i| \le M\{h^4 + h_{i+1}^2[(h_{i+1} - h_i)\varepsilon^{-3} + h_{i+1}^2\varepsilon^{-4}]v_\varepsilon(x_{i-1})\}.$$

Then by distinguishing cases

$$t_{i-1} \ge \alpha$$

and

$$t_{i-1} < \alpha \text{ and } t_{i-1} \le q - 3h,$$

we can prove (3.4) in the same way as in cases I.1 and I.2, respectively. If

$$q - 3h < \alpha < t_{i-1},$$

(3.4) follows from

(3.12) $$|r_i| \le \varepsilon^2 2 \max_{x_{i-1} \le x \le x_{i+1}} |u_\varepsilon''(x)|$$

in the same way as in case I.3.

IIb. Let $b_i^- < 0$, i.e.

$$h_i^2 + h_i h_{i+1} < h_{i+1}^2,$$

which implies:

(3.13) $$\sqrt{2}\lambda'(t_{i-1}) < \lambda'(t_{i+1}).$$

IIb.1 Let $t_{i-1} \ge \alpha$. Then (3.13) reduces to

$$\sqrt{2}\pi'(t_{i-1}) < \pi'(t_{i+1})$$

which means that

$$(\sqrt{2} - 1)[3\omega s^2 + \mu''(\alpha)s + \mu'(\alpha)] \le 12\omega(hs + h^2) + 2\mu''(\alpha)h,$$

where

$$s := t_{i-1} - \alpha \ge 0.$$

From this we get:

$$(\sqrt{2} - 1)\mu'(\alpha) \le Mh,$$

i.e.

$$\varepsilon^{\frac{1}{3}} \le Mh.$$

Now (3.4) follows from (3.11) in the same way as in case I.1.

IIb.2 Let $t_{i-1} < \alpha$ and $t_{i-1} < q - 3h$. Since $t_{i+1} < q$, from (3.9) for $k = 1$ and (3.13) we have

$$\sqrt{2}\mu'(t_{i-1}) < \mu'(t_{i+1}),$$

i.e.

$$\sqrt{2}(q - t_{i+1})^2 < (q - t_{i-1})^2,$$

which is equivalent to

$$t_{i-1} > q - 2^{\frac{5}{4}} \frac{h}{2^{\frac{1}{4}} - 1}.$$

Then we treat this case as I.3 (not I.2 !) and prove (3.4) from (3.12).

IIb.3 Let $q - 3h < t_{i-1} < \alpha$. This is the same case as I.3 and (3.4) follows from (3.12) by the same technique.

Thus, (3.4) is proved for $i = 1(1)m - 1$. When $i = m + 1(1)n - 1$, (3.4) follows in the same way because of the symmetry of the mesh and the estimates (2.1) with respect to $x = \frac{1}{2}$. The similar technique can be used for $i = m$ as well. Note that the fact that λ'' is discontinuous at $x = \frac{1}{2}$ does not have any effect on the proof since $h_m = h_{m+1}$. □

Remark 2. The sufficiently small ε^* required in Theorem 2 is the same one as in Lemma 1 (the same ε^* is needed in Theorem 1 as well), and such that $\varepsilon^* < \frac{1}{8}$ (see the definition of λ), and that (3.10) holds. However, according to Remark 1, if $z \equiv u^*$ and $y \equiv u_*$ (that is, if $z^h = u^* e^h$ and $y^h = u_* e^h$), we would need only $\varepsilon^* < \frac{1}{8}$ (which is not a serious restriction) and (3.10), but then, for a given ε^*, (3.10) can be regarded as a condition on the mesh generating functon parameters a and q.

4. Numerical Results

In order to compare our method to the method from [6], we shall consider the same linear test problem:

$$(4.1) \qquad -\varepsilon^2 u'' + u - 1 = 0, \quad u(0) = u(1) = 0,$$

for which the exact solution u_ε is known. Also, we shall consider a nonlinear problem from [2, pp. 166-168]:

$$-\varepsilon^2 u'' + \frac{u-4}{5-u} = 0, \quad u(0) = u(1) = 0, \tag{4.2}$$

written down with homogeneous boundary conditions. This problem models the biological Michaelis-Menten process without inhibition. Here we have $u^* = 4$ and $u_* = 0$, thus $c^* = 1$ and $c_* = \frac{1}{25}$. Note that (1.3b) is not satisfied.

Let

$$E_h = \|w_\varepsilon^h - u_\varepsilon^h\|,$$

where in case of problem (4.2) we shall replace u_ε^h by the numerical solution on the mesh (2.2) with $n = 512$. We shall always use the mesh (2.2) with $a = 1$ and $q = 0.48$. By changing these parameters, it is possible to change the percentage of the mesh points lying in the layers, cf.[5-9]. We shall also calculate the experimental order of convergence:

$$\text{Ord}_h = \frac{\ln E_h - \ln E_{\frac{h}{2}}}{\ln 2}.$$

The results for problems (4.1) and (4.2) are given in Tables 1 and 2, respectively.

Table 1. E_h and Ord_h for problem (4.1)

n	32	64	128	256	
	2.14(-2)	5.50(-3)	1.24(-4)	7.75(-6)	E_h
$\varepsilon = 2^{-8}$	1.96	5.47	4.00	–	Ord_h
$\varepsilon = 2^{-k}$,	1.79(-3)	1.33(-4)	9.17(-6)	5.69(-7)	E_h
$k = 16(8)48$	3.75	3.86	4.01	–	Ord_h

Table 2. E_h and Ord_h for problem (4.2)

n	32	64	128	256	
$\varepsilon = 2^{-k}$,	2.69(-2)	9.19(-4)	8.26(-5)	4.86(-6)	E_h
$k = 16(8)48$	4.87	3.48	4.09	–	Ord_h

These results confirm the fourth order uniform convergence, obtained theoretically. The results in Table 1 are worse for $\varepsilon = 2^{-8}$ than the corresponding results from [6], but for other values of ε they are even slightly better than in [6]. The number of the mesh points lying in $[0, \varepsilon]$, is the same as in [6]. It is denoted by n_ε and shown in Table 3, where we also present the number of the mesh points at which T_C is used; this number is denoted by n_C. The number n_C is much greater in our method than in the method from [6], nevertheless the fourth order uniform accuracy is retained.

Table 3. n_ε and n_C for $\varepsilon = 2^{-32}$

n	32	64	128	256	
n_ε	8	16	31	62	[6] and our method
n_C	4	6	4	6	[6]
	9	11	15	23	our method

Finally, let us note that our method, in the same way as the method from [6], can be applied to other semilinear singular perturbation problems which do not satisfy the hypotesis H1 and H2, but their solutions behave as described by Lemma 2.

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REZIME

O NUMERIČKOM REŠAVANJU SEMILINEARNIH SINGULARNIH PERTURBACIONIH PROBLEMA KORIŠČENJEM HERMITOVE ŠEME

Numerički se rešava semilinearni singularno perturbovani konturni problem pomoću metoda konačnih razlika koji koristi kombinaciju Hermitove i standardne centralne šeme na specijalnoj neekvidistantnoj mreži. Metod predstavlja modifikaciju postupka iz [6]. Dokazan je isti rezultat (četvrti red tačnosti, uniformno po perturbacionom parametru), ali bez ograničenja nelinearnosti koji je korišćen u [6].

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