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LAYER FUNCTIONS AND SPECTRAL APPROXIMATION

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Abstract

The two-point boundary layer problem, described by the second order differential equation with coefficientes depending on the small perturbation parameter, is considered. The approximate solution is constructed as a sum of the layer function and a truncated orthogonal series, using the asymptotic expansion of the first order. The theoretical results are illustrated by two numerical examples.

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1. Introduction

We shall consider the boundary layer problem

$$Ly \equiv -\varepsilon y''(x) + f(x,\varepsilon)y'(x) + g(x,\varepsilon)y(x) = h(x,\varepsilon),\ x \in [0,1] \tag{1.1}$$

$$y(0) = \alpha,\ y(1) = \beta, \tag{1.2}$$

where $\varepsilon \in I = (0, \varepsilon_0)$, $0 < \varepsilon_0 \ll 1$ is a small parameter and $f, g, h \in C^2(\Omega)$, with $\Omega = [0,1] \times I$.

Let us suppose that one of the following ceses holds

1° $f(x,\varepsilon) \geq M > 0,\ g(x,\varepsilon) \geq 0$

2° $f(x,\varepsilon) \equiv 0,\ g(x,\varepsilon) \geq K^2 > 0,$

for all $x \in [0,1]$ and each ε sufficiently small. These conditions provide the existence and the inverse monotonicity of the unique solution $y(x) \in C^2[0,1]$ of the problem (1.1),(1.2).

In the case 1° we have a nonselfadjoint boundary value problem and its solution displays a boundary layer at the right endpoint, and in the case 2° we have a selfadjoint problem and its solution, in general, displays two boundary layers, one at each endpoint.

In the first part of this paper we shall construct the layer functions for both cases and transform the original problem. In the second part we shall determine the orthogonal projection of the unknown solution and state the main theorem which enables us to evaluate the coefficients of the spectral approximation, using the collocation method. In the third part we shall give the error estimate and two numerical examples.

2. Transformation of the problem

Let us represent the functions f, g and h in the form of the asymptotic power series

$$\begin{bmatrix} f(x,\varepsilon) \\ g(x,\varepsilon) \\ h(x,\varepsilon) \end{bmatrix} = \sum_{m=0}^{\infty} \begin{bmatrix} f_m(x) \\ g_m(x) \\ h_m(x) \end{bmatrix} \varepsilon^m. \tag{2.1}$$

In the case 1° $f(x,\varepsilon) \geq M > 0$ for sufficiently small ε implies $f_0(x) > 0$ and the solution of the reduced problem

$$f_0(x)y_R'(x) + g_0(x)y_R(x) = h_0(x),\ x \in [0,1],\ y_R(0) = \alpha \tag{2.2}$$

can be exactly evaluated. In the case 2° $g(x,\varepsilon) \geq K^2 > 0$ implies $g_0(x) > 0$ and the reduced solution is $y_R(x) = \frac{h_0(x)}{g_0(x)}$.

In the first case, according to [3], the layer function is of the form

$$e(x) = \beta_0 e^{\eta(x)} \tag{2.3}$$

with

$$\beta_0 = \beta - y_R(1) \tag{2.4}$$

and

$$\eta(x) = -\frac{1}{\varepsilon}\int_x^1 f_0(t)dt. \tag{2.5}$$

In the second case we have two layer functions, one for each endpoint:

$$e_1(x) = \delta_0 e^{\xi(x)},\ e_2(x) = \gamma_0 e^{\nu(x)}, \tag{2.6}$$

with the functions

$$\delta_0 = \alpha_0 \sqrt[4]{\frac{g_0(0)}{g_0(x)}}, \alpha_0 = \alpha - \frac{h_0(0)}{g_0(0)}, \gamma_0 = \beta_1 \sqrt[4]{\frac{g_0(1)}{g_0(x)}}, \beta_1 = \beta - \frac{h_0(1)}{g_0(1)}. \tag{2.7}$$

and

$$\xi(x) = \frac{1}{\sqrt{\varepsilon}}\int_0^x \sqrt{g_0(t)}dt,\ \nu(x) = \frac{1}{\sqrt{\varepsilon}}\int_1^x \sqrt{g_0(t)}dt. \tag{2.8}$$

Here, we shall denote $e(x) = e_1(x) + e_2(x)$.

Using these layer functions we can transform original problem (1.1),(1.2). The idea is to represent the solution in the form

$$\tilde{y}(x) = e(x) + u(x) \tag{2.9}$$

where $u(x)$ is the solution of the following problem:

$$-\varepsilon u'' + \tilde{f}(x)u' + \tilde{g}(x)u = \varphi(x),\ x \in [0,1] \tag{2.10}$$

$$u(0) = A,\ \ u(1) = B, \tag{2.11}$$

with

$$\tilde{f}(x) = f_0(x) + \varepsilon f_1(x),\ \tilde{g}(x) = g_0(x) + \varepsilon g_1(x) \tag{2.12}$$

and

$$\begin{aligned}\varphi(x) = \ & h_0(x) + \varepsilon h_1(x) + (f_0'(x) - f_0(x)f_1(x) - \\ & -\varepsilon f_0(x)f_2(x) - g_0(x) - \varepsilon g_1(x))\, e(x),\end{aligned} \tag{2.13}$$

$$A = \alpha - \beta_0 e^{-\frac{1}{\varepsilon}\int_0^1 f_0(t)dt},\ B = y_R(1) \tag{2.14}$$

for the nonselfadjoint problem, and

$$\varphi(x) = h_0(x) + \varepsilon h_1(x) + (\frac{g_0'(x)}{2\sqrt{g_0(x)}}\sqrt{\varepsilon} - \varepsilon g_1(x))e(x), \tag{2.15}$$

$$A = y_R(0) - \frac{\beta_1}{k},\ B = y_R(1) - \alpha_0 k,\ k = k_2 e^{k_1}, \tag{2.16}$$

$$k_1 = \frac{1}{\bar{\varepsilon}} \int_0^1 \sqrt{g_0(t)}dt,\ k_2 = \sqrt[4]{\frac{g_0(0)}{g_0(1)}},$$

for the selfadjoint problem.

Remark 1. Since equation (2.10) is obtained by rejecting terms containing ε^i, for $i - 2$, (2.9) represents the approximation to the exact solution $y(x)$ of the order $O(\varepsilon^2)$.

3. Orthogonal projecting

Let σ_n denote the projecting operator, such that

$$\sigma_n\ :\ u(x) -\ u_n(x) = \sum_{k=0}^{n} {}^{-}a_k Q_k(x), \tag{3.1}$$

where $-Q_k$, $k = 0, 1, \dots, n-$are classical orthogonal polynomials upon [0,1]. The notation $^{-}$means that the summation involves the term for $k = 0$ multiplied by $\frac{1}{2}$.

As the elements of the orthogonal basis we shall choose shifted Chebyshev polynomials $T_k^-(x)$, orthogonal upon the interval [0,1] with respect to the weight function

$$p(x) = \frac{1}{2\sqrt{x(1-x)}}.$$

They represent a particular solution of the differential equation

$$x(1-x)y^{-}- (x - \frac{1}{2})y^{-} + k^2 y = 0,\ k = 0, 1, \dots$$

and can be determined by Bonnet's recurrence relation

$$T_{k+1}^-(x) = (4x-2)T_k^-(x) - T_{k-1}^-(x),\ k = 1, \dots, \tag{3.2}$$

$$T_0^-(x) = 1,\ T_1^- = 2x - 1.$$

It is well known that polynomials T_k^- form the basis of the Hilbert space

$$L[0,1] - -f :\ [0,1] -\ R, \text{ mesurable such that } -f - < + - -$$

Here, we define the inner product and norm as

$$(f,g)=\int_0^1 f(x)g(x)p(x)dx,\ \|f\|^2=(f,f)$$

and for each $i,j\in\mathbf{N}$ it is valid that $(T_i^*,T_j^*)=\pi\delta_{i,j}$.

So, when speaking of the spectral approximation for the solution $u(x)$ of the problem (2.10),(2.11), we, in fact, want to find its orthogonal projection in terms of the definition (3.1). For that purpose we are going to use the collocation method, which means that we look for $u_n\in\mathbf{P}^n$,such that

$$u_n(0)=A,\ u_n(1)=B \tag{3.3}$$

and

$$-\varepsilon u_n''(x_i)+\tilde{f}(x_i)u_n'(x_i)+\tilde{g}(x_i)u_n(x_i)=\varphi(x_i),\ i=1,\dots,n-1, \tag{3.4}$$

where we use the Gauss-Lobato collocation nodes

$$x_i=\frac{1}{2}(\cos\frac{i\pi}{n}+1),\ i=1,\dots,n-1. \tag{3.5}$$

Here $\mathbf{P}^n$ denotes the space of all the real polynomials of degree up to n.

Now, we can state the theorem which enables us to evaluate the coefficients of the orthogonal projection $u_n(x)$ in (3.3),(3.4).

Theorem 1. *The coefficients a_k of the spectral approximation*

$$u_n(x)=\sum_{k=0}^{n} a_k T_k^*(x) \tag{3.6}$$

for the solution $u(x)$ of the problem (2.10),(2.11) represent the solution of the system

$$\sum_{k=0}^{n} f_{k,i}a_k=\varphi_i,\ i=0,\dots,n \tag{3.7}$$

with

$$f_{k,0}=(-1)^k,\ f_{k,n}=1,\ k=0,\dots,n,\ \varphi_0=A,\ \varphi_n=B \tag{3.8}$$

$$\begin{cases} f_{k+1,i} &= (4x_i-2)f_{k,i}-f_{k-1,i}+4\tilde{f}(x_i)T_k^*(x_i)+\\ &\quad +2k\sum_{j=k-1(2)}^{0} T_j^*(x_i)\\ \varphi_i &= \varphi(x_i),\ i=1,\dots,n-1,\ k=1,\dots,n-1\\ f_{0,i} &= \tilde{g}(x_i),\ f_{1,i}=2\tilde{f}(x_i)+(2x_i-1)\tilde{g}(x_i),\\ &\quad i=1,\dots,n-1, \end{cases} \tag{3.9}$$

where the points x_i, are given by (3.5).

Proof. We substitute (3.6) into (3.3) and (3.4). Since

$$T_k^{*}(0) = (-1)^k, \;\; T_k^{*}(1) = 1,$$

(3.3) gives the first and the last equation in the system (3.7) with notation (3.8). In order to obtain the other $n-1$ equations, whose coefficients are determined recurrently by (3.9), we have to start from (3.2). Deriving it, we come to

$$\begin{aligned} T_{k+1}^{*\prime}(x) &= (4x-2)T_k^{*\prime}(x) - T_{k-1}^{*\prime}(x) + 4T_k^{*}(x) \\ T_{k+1}^{*\prime\prime}(x) &= (4x-2)T_k^{*\prime\prime}(x) - T_{k-1}^{*\prime\prime}(x) + 8T_k^{*\prime}(x). \end{aligned} \tag{3.10}$$

The substitution of (3.6) into (3.4) gives that the coefficients at a_{k+1}, for $k = 1, \ldots, n$, are

$$f_{k+1,i} = -\varepsilon T_{k+1}^{*\prime\prime}(x_i) + \tilde{f}(x_i)T_{k+1}^{*\prime}(x_i) + \tilde{g}(x_i)T_{k+1}^{*}(x_i), \tag{3.11}$$

$$\varphi_i = \varphi(x_i), \;\; i = 1, \ldots, n.$$

After the use of (3.2) and (3.10) we have

$$\begin{aligned} f_{k+1,i} \;\; = \;\; & -\varepsilon(4x_i - 2)T_k^{*\prime\prime}(x_i) + \varepsilon T_{k-1}^{*\prime\prime}(x_i) - 8\varepsilon T_k^{*\prime}(x_i) + \\ & + \tilde{f}(x_i)(4x_i - 2)T_k^{*\prime}(x_i) - \tilde{f}(x_i)T_{k-1}^{*\prime}(x_i) + 4\tilde{f}(x_i)T_k^{*}(x_i) + \\ & + \tilde{g}(x_i)(4x_i - 2)T_k^{*}(x_i) - \tilde{g}(x_i)T_{k-1}^{*}(x_i), \; i = 1, \ldots, n-1. \end{aligned}$$

According to the notation (3.11) this gives

$$f_{k+1,i} = (4x_i - 2)f_{k,i} - f_{k-1,i} - 8\varepsilon T_k^{*\prime}(x_i) + 4\tilde{f}(x_i)T_k^{*}(x_i), \tag{3.12}$$

$$i = 1, \ldots, n-1.$$

Since

$$T_k^{*\prime}(x) = 2k \sum_{j=k-1(2)}^{0} {}' T_j^{*}(x)$$

(3.12) finally gives us the first equality in (3.9). The notation $j = k - 1(2)$ means that the summation involves only the terms with indices $k-1, k-3, \ldots$ up to 0 or 1. The last two equalities in (3.9) are obtained derectly from (3.11), which, when $k = -1$, gives

$$T_0^{*\prime\prime}(x_i) = 0, \; T_0^{*\prime}(x_i) = 0, \; T_0(x_i) = 1, \; i = 1, \ldots, n-1,$$

and, when $k = 0$,

$$T_1''(x_i) = 0,\ T_1'(x_i) = 2,\ T_1(x_i) = 2x_i - 1,\ i = 1, \ldots, n-1.$$

For the evaluation of $T_k(x_i)$ in (3.9) one has to use the recurrence relation (3.2). □

Remark 2. The same procedure can be carried out using a larger number of terms in asymptotic expansion.

4. Special cases

If we consider a special case of the differencial equation (1.1) where the coefficients do not depend on ε, i.e. if we have the problem

$$Ly = -\varepsilon y''(x) + f(x)y'(x) + g(x)y(x) = h(x),\ x \in [0,1], \tag{4.1}$$

with the boundary conditions (1.2), we can determine the layer functions (2.3) and (2.6) in a simpler way, which enables us to avoid the calculation of the integrals (2.5) and (2.8). These constructions are proposed in [1]. In the case of the nonselfadjoint problem, in the layer function (2.3) we take

$$\eta(x) = \frac{(x-1)f(1)}{\varepsilon}, \tag{4.2}$$

and in the selfadjoint case in (2.6) we take

$$\begin{aligned} &\delta_0 = \alpha - y_R(0),\ \xi(x) = -\sqrt{\tfrac{g(0)}{\varepsilon}}x,\ \gamma_0 = \beta - y_R(1), \\ &\nu(x) = -\sqrt{\tfrac{g(1)}{\varepsilon}}(1-x). \end{aligned} \tag{4.3}$$

Using the same procedure for the transformation of the original problem we shall come, again to the problem (2.10),(2.11) with

$$\tilde{f}(x) = f(x),\ \tilde{g}(x) = g(x) \tag{4.4}$$

and

$$\varphi(x) = h(x) - (g(x) - \frac{f(1)(f(1) - f(x))}{\varepsilon})e(x) \tag{4.5}$$

$$A = \alpha - \beta_0 e^{-\frac{f(1)}{\varepsilon}},\ B = y_R(1) \tag{4.6}$$

for the nonselfadjoint problem, and

$$\varphi(x) = h(x) - (g(x) - g(0))e_1(x) - (g(x) - g(1))e_2(x) \tag{4.7}$$

$$\begin{aligned} A &= y_R(0) - (\beta - y_R(1))e^{-\sqrt{\frac{g(1)}{\varepsilon}}}, \\ B &= y_R(1) - (\alpha - y_R(0))e^{-\sqrt{\frac{g(0)}{\varepsilon}}} \end{aligned} \tag{4.8}$$

for the selfadjoint problem.

Let us notice that the term (2.9) will now represent the exact solution to the problem (4.1),(1.2). Using the above notation the system for evaluation of the coefficients of the spectral approximation (3.6) is again given by the Theorem 1.

5. The error estimate

Since there is no method for an exact error estimate in the case of spectral approximations, we are going to use an approximate error estimate proposed in [2]. It is known that when $n \to \infty$ the spectral approximation (3.6) of the function $u(x)$ tends to $u(x)$. Thus, it is necessary to increase n until the values for the coefficients a_k, evaluated for $n-1$ and n, become sufficiently close. Then, we can suppose that u_{2n-1} (evaluated using $2(n-1)$ collocation points) sufficiently well approximates the exact solution and we can write

$$\|u(x) - u_n(x)\| \approx \|u_{2n-1}(x) - u_n(x)\|. \tag{5.1}$$

Let us denote by

$$y_n(x) = e(x) + u_n(x) = e(x) + \sum_{k=0}^{n} \bar{a}_k \bar{T}_k(x) \tag{5.2}$$

$$y_{2n-1}(x) = e(x) + u_{2n-1}(x) = e(x) + \sum_{k=0}^{2n-1} \bar{b}_k \bar{T}_k(x) \tag{5.3}$$

the approximations to (2.9), where $u_n(x)$ and $u_{2n-1}(x)$ are spectral approximations of the problem (2.10),(2.11). For the estimate of the error function

$$d(x) = \|y(x) - y_n(x)\|, \tag{5.4}$$

where $y(x)$ is the exact solution to the problems (4.1),(1.2) we can prove the following theorem:

Theorem 2. *For the problem (4.1),(1.2) we have*

$$
(5.5)\qquad |d(x)| \le \sum_{k=0}^{n} |b_k - a_k| + \sum_{k=n+1}^{2n-1} |b_k| .
$$

Proof. For the problem (4.1),(1.2) the exact solution $y(x) = \tilde{y}(x)$ is given by (2.9) Thus,

$$
d(x) = |\tilde{y}(x) - y_n(x)| = |u(x) - u_n(x)| .
$$

By (5.1)

$$
(5.6)\quad d(x) \le |u_{2n-1}(x) - u_n(x)| = \Big|\sum_{k=0}^{2n-1} b_k T_k^*(x) - \sum_{k=0}^{n} a_k T_k^*(x)\Big| \le
$$

$$
\le \sum_{k=0}^{n} |b_k - a_k| |T_k^*(x)| + \sum_{k=n+1}^{2n-1} |b_k| |T_k^*(x)| .
$$

Using the fact that $|T_k^*(x)| \le 1$ for all $x \in [0,1]$ and $k = 0, 1, \ldots$ the above relation gives (5.5).

In the case of problems (1.1),(1.2), the estimate (5.5) becomes

$$
(5.7)\qquad d(x) \le \sum_{k=0}^{n} |b_k - a_k| + \sum_{k=n+1}^{2n-1} |b_k| + M\varepsilon^2,
$$

because of Remark 1.

6. Numerical examples

As the first example we shall consider the nonselfadjoint boundary layer problem

$$
(6.1)\qquad -\varepsilon y'' + \frac{2+4\varepsilon-2\varepsilon x}{(2-x)^2} y' = \frac{2\pi}{(2-x)^4}\sin\frac{\pi(1-x)}{2-x} +
$$

$$
\frac{\varepsilon\pi}{(2-x)^3}\cos\frac{\pi(1-x)}{2-x},\ y(0) = y(1) = 0,
$$

given in [4]. According to formulas (2.3)-(2.5), the layer function is

$$
e(x) = -e^{-\frac{2}{\varepsilon}\frac{1-x}{2-x}} .
$$

In the following tables we shall give order of the error (5.6) for different values of n at several points of boundary layer and out of the boundary layer.

$\varepsilon = 10^{-6}$					
x	y	n=3	n=5	n=7	n=9
out the layer		E(-2)	E(-3)	E(-5)	E(-6)
0.999995	0.99995	E(-6)	E(-7)	E(-9)	E(-10)
0.999999	0.86	E(-7)	E(-8)	E(-9)	E(-10)
0.9999995	0.63	E(-7)	E(-8)	E(-10)	E(-10)
0.9999998	0.33	E(-8)	E(-8)	E(-10)	E(-10)
0.9999999	0.18	E(-8)	E(-9)	E(-10)	E(-10)
0.99999999	0.02	E(-9)	E(-10)	E(-10)	E(-10)

Table 1.

$\varepsilon = 10^{-8}$						
x		n=3	n=5	n=7	n=9	n=13
out the layer		E(-2)	E(-3)	E(-5)	E(-6)	E(-8)
0.99999995	0.99995	E(-8)	E(-9)	E(-10)	E(-10)	E(-10)
0.99999999	0.86	E(-9)	E(-10)	E(-10)	E(-10)	E(-10)
0.999999995	0.63	E(-9)	E(-10)	E(-10)	E(-10)	E(-10)
0.999999998	0.33	E(-10)	E(-10)	E(-10)	E(-10)	E(-10)
0.999999999	0.18	E(-10)	E(-10)	E(-10)	E(-10)	E(-10)
0.9999999999	0.02	E(-10)	E(-10)	E(-10)	E(-10)	E(-10)

Table 2.

For the second example we shall take the selfadjoint problem

(6.2) $$-\varepsilon y^{-} + y = \cos^2 \pi x - 2\varepsilon\pi^2 \cos 2\pi x,\ y(0) = y(1) = 0$$

from [1]. According to (2.6) and (4.2)

$$e(x) = e^{-\frac{x}{\sqrt{\varepsilon}}} + e^{\frac{x-1}{\sqrt{\varepsilon}}}.$$

In the following tables we give the order of the error (5.6) at several points of the left-end boundary layer and out of the layer.

$\varepsilon = 10^{-6}$				
x	y	n=5	n=8	n=12
0.0001	-0.095	E(-5)	E(-7)	E(-9)
0.0003	-0.25	E(-4)	E(-7)	E(-9)
0.0008	-0.55	E(-4)	E(-6)	E(-9)
0.0015	-0.78	E(-4)	E(-6)	E(-9)
0.0025	-0.92	E(-3)	E(-6)	E(-9)
0.005	-0.993	E(-3)	E(-6)	E(-9)
0.001	-0.999	E(-3)	E(-5)	E(-9)
out of layer		E(-2)	E(-5)	E(-8)

Table 3.

$\varepsilon = 10^{-12}$				
x	y	n=5	n=8	n=12
0.0000001	-0.095	E(-8)	E(-9)	E(-9)
0.0000003	-0.25	E(-7)	E(-9)	E(-9)
0.0000008	-0.55	E(-7)	E(-9)	E(-9)
0.0000015	-0.78	E(-7)	E(-9)	E(-9)
0.0000025	-0.92	E(-6)	E(-8)	E(-9)
0.000005	-0.993	E(-6)	E(-8)	E(-9)
0.000001	-0.999	E(-6)	E(-8)	E(-9)
out of layer		E(-2)	E(-5)	E(-8)

Table 4.

$\varepsilon = 10^{-18}$				
x	y	n=5	n=8	n=12
0.0000000001	-0.095	E(-9)	E(-9)	E(-9)
0.0000000003	-0.25	E(-9)	E(-9)	E(-9)
0.0000000008	-0.55	E(-9)	E(-9)	E(-9)
0.0000000015	-0.78	E(-9)	E(-9)	E(-9)
0.0000000025	-0.92	E(-9)	E(-9)	E(-9)
0.000000005	-0.993	E(-9)	E(-9)	E(-9)
0.000000001	-0.999	E(-9)	E(-9)	E(-9)
out of layer		E(-2)	E(-5)	E(-9)

Table 5.

These numerical examples show the high accuracy of the proposed method using only a small number of terms in the appropriate orthogonal series.

It is also significant that these results are better than those obtained by the use of the first two terms in asymptotic solutions for the given problems.

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REZIME

FUNKCIJE SLOJA I SPEKTRALNE APROKSIMACIJE

Posmatra se konturni problem opisan diferencijalnom jednačinom drugog reda, čiji koeficijenti zavise od malog perturbacionog parametra. Približno rešenje je konstruisano kao zbir funkcije sloja i parcijalne sume ortogonalnog reda, koristeći asimptotski razvoj prvog reda. Teorijski rezultati su ilustrovani na dva numerička primera.

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