Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 23, 2 (1993), 345 - 350

Review of Research
Faculty of Science
Mathematics Series

COLLOCATION SPLINE METHODS IN SOLVING BOUNDARY VALUE PROBLEMS

Katarina Surla
Institute of Mathematics, University of Novi Sad,
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Vladimir Jerković
Radio Television Novi Sad
Žarka Zrenjanina 3, 21000 Novi Sad, Yugoslavia

Abstract

The spline difference schemes for the problem: $-\varepsilon y'' + p(x)y = f(x), y(0) = \alpha_0, y(1) = \alpha_1$, are considered. The error estimates have the optimal form $O(h^4/(\varepsilon + h^2))$. The different ways for derivation of the schemes and different possibilities for the global approximations are presented.

AMS Mathematics Subject Classifications (1991): 65L10
Key words and phrases: Spline function, collocation, difference scheme, singular perturbation problem.

Let us consider the following singularly perturbed problem

$$(1) \qquad \begin{cases} Ly = -\varepsilon y'' + p(x)y = f(x), \quad x \in I = [0,1], \\ y(0) = \alpha_0, y(1) = \alpha_1 \end{cases}$$

It is known that the solution of the problem exibits boundary layers near to 0 and near to 1 and that the classical cubic spline methods do not converge. Because of that some modiffications are necessary. Namely, we seak the solution in the form of the cubic spline $e(x)$ on equidistant mesh $x_i = i * h, i = 0(1)n+1, h = 1/(n+1)$,

(2) $$e(x) = \sum_{i=-3}^{n-1} b_i B_i(x)$$

where

$$B_i(x) = 4\sum_{p=i}^{i+4} \frac{(x_p - x)_+^3}{\omega_i'(x_p)}, \quad \omega_i(t) = \prod_{j=i}^{i+4}(t - x_j).$$

Spline $e(x)$ is a polynomial of degree three on $[x_j, x_{j+1}]$. The collocation conditions we modify by using free parameter σ which we determine in order to come nearer to the exact solution:

(3) $$\sigma_j e''(x) + p^- e(x) = f^-, \quad x = x_j, x = x_{j-1},$$

(4) $$\sigma_j e''(x) + p^+ e(x) = f^+, \quad x = x_j, x = x_{j+1},$$

where p^- and f^- are constant aproximations to $p(x)$ and $f(x)$ for $x \in [x_{j-1}, x_j]$ and similarly p^+ and f^+ are constant aproximations to $p(x)$ and $f(x)$ on the interval $[x_j, x_{j+1}]$ for fixed j. The choice of approximations to $p(x)$ and $f(x)$ determines the particular scheme. The parameter σ we determine according to the following theorem.

Theorem 1. *([1]) Let $p(x)$ and $f(x) \in C^4[0,1]$ and $p'(0) = p'(1) = 0$. Then $y(x) = gu(x) + qw(x) + c(x)$ where $|g|, |q| \le M$, $v(x) = exp(-x\sqrt{p(0)/\varepsilon})$, $w(x) = exp(-(1-x)\sqrt{p(1)/\varepsilon})$, $|c^{(i)}(x)| \le M(1 + \varepsilon^{1-i/2})$, i=1,2,3,4.*

M denotes different constants independent of ε and h.

The fitting factor σ we determine so as the truncation errors for the functions $v(x)$ and $w(x)$ will be zero in the case $p(x) = p = const$.

In [7] it has been is shown that in the case $p^+ = p^- = p_j = p(x_j)$ we obtain the scheme from [1]:

(5) $$\sigma_j(u_{j-1} - 2u_j + u_{j+1})/h^2 + p_j u_j = f_j$$

$$u_0 = \alpha_0, \; u_{n+1} = \alpha_1.$$

When we use piecewise linear approximations to $p(x)$ and $f(x)$ we obtain the scheme which can be derived from the consistency relation

(6) $$e''_{j-1}/6 + 2e''_j/3 + e''_{j-1} = (e_{j-1} - 2e_j + e_{j+1})/h^2$$

by replacing the second derivatives from the collocation conditions:

(7) $$e''_j = (p_j e_j - f_j)/\sigma_j.$$

The corresponding spline belongs to $C^2[0,1]$ and

$$\sigma_j = h^2 p_j/6(1 + 3/(2\sinh^2(h/2\sqrt{p_j/\varepsilon}).$$

The error estimate $O(h^2)$ is obtained. In [2] the analyzed cubic spline $Q(x)$ was of the form:

$$Q(x) = \sum_{i=-3}^{n} a_i P(x/h - i + 1/2),$$

where

$$P(x) = \frac{1}{6}\sum_{i=0}^{n}(-1)^i \begin{pmatrix} 4 \\ i \end{pmatrix} (x-i)_+^3,$$

$$(x-t)_+ = \begin{cases} x - t & \text{for} \quad x \geq t \\ 0 & \text{for} \quad x < t. \end{cases}$$

The spline is a polynomial of degree three on $[(i-1/2)h, (i+1/2)h]$ and the consistency relations

$$1/24(Q''_{j-1} + 22Q''_j + Q''_{j+1}) = h^2(Q_{j-1} - 2Q_j + Q_{j+1})$$

was derived. Using the collocation conditions

$$Q''_j = (p_j Q_j - f_j)/\sigma_j.$$

we obtain the scheme

(8) $$RQ_j = Gf_j, j = 1(1)n;\ u_0 = \alpha_0,\ u_{n+1} = \alpha_1.$$

where

$$Ru_j = (\frac{p_{j-1}}{24\sigma_{j-1}} - \frac{1}{h^2})u_{j-1} + (\frac{11p_j}{12\sigma_j} + \frac{2}{h^2})u_j + (\frac{p_{j+1}}{24\sigma_{j+1}} - \frac{1}{h^2})u_{j+1}$$

$$Gf_j = \frac{f_{j-1}}{24\sigma_{j-1}} + \frac{11f_j}{12\sigma_j} + \frac{f_{j+1}}{24\sigma_{j+1}}.$$

The fitting factor for this scheme in the case $p(x) = p = const.$ has the form $\sigma(\rho) = \frac{ph^2(\exp(\rho)+22+\exp(-\rho))}{24(\exp(\rho)-2+\exp(-\rho))}$, $\rho = h\sqrt{p/\varepsilon}$. When $p(x) \neq const$ we define $\sigma_j = \sigma(\rho_j)$ with $\rho_j = h\sqrt{p_j\varepsilon}$. By using technique from [7] or [8] we can prove that, under the conditions of Theorem 1,

$$|Q_j - y(x_j)| \leq Mh^2, j = 1(1)n.$$

Taking into account that ([3] and [8]):

$$y''(x_j) - e_j'' = 1/12h^2y_j^{(4)} + O(h^4),$$

$$y''(x_j) - Q_j'' = -1/24h^2y_j^{(4)} + O(h^4),$$

$$y_j'' - \frac{y_{j-1} - 2y_j + y_{j-1}}{h^2} = -1/12h^2y_j^{(4)} + O(h^4),$$

and using linear combinations of schemes (5), (6)(7) and (8) we obtain the schemes having the higher order of the classical convergence.

The linear combination of two first schemes was considered in [8] and the error estimate $O(h^4/(\varepsilon+h^2)$ was proved. Now, we form a linear combination of the scheme (5) and scheme (8). Namely, if we multiply scheme (8) by two and then subtract scheme (5) we obtain a new scheme:

$$Au_j = Bf_j, \tag{9}$$

where

$$Au_j = (\frac{p_{j-1}}{12\sigma_{j-1}} - \frac{1}{h^2})u_{j-1} + (\frac{10p_j}{12\sigma_j} + \frac{2}{h^2})u_j + (\frac{p_{j+1}}{12\sigma_{j+1}} - \frac{1}{h^2})u_{j+1}$$

$$Bf_j = \frac{f_{j-1}}{12\sigma_{j-1}} + \frac{10f_j}{12\sigma_j} + \frac{f_{j+1}}{12\sigma_{j+1}}.$$

The fitting factor, determined in the mentioned way, has the form

$$\sigma(\rho) = \frac{ph^2(\exp(\rho) + 10 + \exp(-\rho))}{12(\exp(\rho) - 2 + \exp(-\rho))}, \quad \rho = h\sqrt{p/\varepsilon}$$

and $\sigma_j = \sigma(\rho_j)$, $\rho_j = h\sqrt{p_j/\varepsilon}$.

This is the same scheme as the one derived in [8]. Thus the convergence is given by

$$|y_j - u_j| \leq Mh^4/(\varepsilon + h^2)$$

in the case $y \in C^6[0,1]$. An analogus linear combination of schemes (6), (7) and (8) leads to the same schemes. Although the schemes have the same nodal form they have the different global extensions, and we can choose either the first or the second form according to our purpose. In the first case the corresponding collocation function is the spline from $C^1[0,1]$ with the knots $x_j, j = 0(1)n+1$. In the second case the spline belongs to $C^2[0,1]$ and it has the knots at the points $x_j, j = 0(1)n+1$ and $x_{j-1/2}, j = 0(1)n$. The numerical results supporting the given estimates have been presented in [8].

References

[1] Doolan, E.P., Miller, J.J.H., Schilders, W.H.A., Uniform numerical methods for problems with initial and boundary layers, Dublin, Boole Press 1980.

[2] Sakai, M., Some New Linear Relations for Odd Degree Polinomial Splines at Mid-Points, Proc. Japan Acad. 59,Ser. A (1983),24–26.

[3] Sakai, M., Usmani, A. R., Some Relations between Cubic Spline Interpolation at Mid-Points and Mesh Points, Math. Japonica, 29, N6 (1984), 909–917.

[4] Surla, K., A uniformly convergent spline difference scheme for a self adjoint singular perturbation problem, Univ. u Novom Sadu Zb. Rad. Prirod. Mat. Fak. Ser. Mat., 17,2(1987), 31–38.

[5] Surla, K., A collocation by spline in tension, Approx. Theory and Its Appl., 6:2(1990),101–110.

[6] Surla, K., Stojanović, M., A solving singularly perturbed boundary value problem by spline in tension, J. Comput. Appl. Math., 24(1988), 355–363.

[7] Surla, K., Uzelac, Z., Families of quadratic and cubic spline difference schemes, Proceedings of the Conference ISAM/89, Spline in Numerical Analysis, Weissig 1989, 171-180.

[8] Surla, K., Uzelac, Z., An optimal uniformly convergent OCI difference scheme for a singular perturbation problem. Intern. J. Comput. Math. 34(1990), No 3+4,239–250.

REZIME

SPLAJN KOLOKACIONE METODE U REŠAVANJU GRANIČNIH PROBLEMA

U [8] izvedena je splajn diferencna šema za problem $-\varepsilon y''+p(x)y = f(x), y(0) = \alpha_0$,
$y(1) = \alpha_1$. Ocena greške je data izrazom $O(h^4/(\varepsilon+h^2))$. U ovom radu, šema je izvedena na više različitih načina, čime su omogućene različite globalne aproksimacije.

Received by the editors September 11, 1990