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CLASSIFICATION OF SOME MODIFICATIONS OF THE PROPOSITIONAL ALGEBRA

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Abstract

In this paper we present the classification of functions of some modifications of the propositional algebra, and the complete classification of corresponding symmetric functions.

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1. Introduction

Let P_2 is the set of functions of the propositional algebra, i.e. the union of all the functions $\{f : \{0,1\}^n \to \{0,1\},\ \ for\ n = 0, 1, 2, \ldots\}$.

A subset F of P_2 is said to be closed if it does not yield a function which is not in F by means of superposition (e.g. see sec. 2) among functions in F.

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A clone C is subset of P_2 containing the projection functions x_i and closed with respect to superposition. A subset of functions in H (for $H \subset P_2$) is complete in H if every function in H can be represented as a superposition of the elements of the set.

For closed sets F and H such that $F \subset H$ (proper inclusion), F is H-maximal set if there is no closed set G such that $F \subset G \subset H$. If the number m of H-maximal sets is finite then a subset of functions is complete in H if and only if it is not contained in any one H-maximal set [6].

Finite complete set of functions in H is called base, if none of its subsets is complete in H. A set of functions $\{f_1, \ldots, f_s\}$, is called pivotal set in H, if and only if for every function f_i, $1 \leq i \leq s$, there exsists at least one H-maximal set in which function f_i is not included and all the other function $f_1, \ldots, f_s$ are included. From these definition follows that the base is complete pivotal set of functions. The rank of base, i.e. *pivotal set*, is the number of elements of the base - *pivotal set.*

For $f \in H$, its characteristic vector is $a_1 a_2 \ldots a_m$, $a_i \in \{0,1\}$, $1 \leq i \leq m$, where $a_i = 0$ if and only if the function is element of the $i-th$ H-maximal set. All functions of some characteristic vector determine the one class of functions. Classes of functions for each base determine the class of given base. Analogously are defined classes of pivotal sets.

We denote each class of functions by its ordinal number, or by $/M_{i_1}, \ldots, M_{i_j}/$, where $M_{i_1}, \ldots, M_{i_j}$ are all maximal sets containing the functions of the class. Thus set consisting of classes $c_1, \ldots, c_j$ (especially, class of bases and pivotal sets) we denote by $(c_1, \ldots, c_j)$.

We obtain the classification of H if we determine all nonempty classes of functions of the set H corresponding classes of bases and pivotal sets.

We use the notation of functions preserving a relation to describe subsets of P_2. An $h-ary$ relation $\underline{X}$ on $E_2 = \{0,1\}$, $h \geq 1$, is a subset of E_2^n whose elements are written as columns, i.e.:

$$\begin{pmatrix} a_1 \\ \vdots \\ a_h \end{pmatrix} \in \underline{X} \Longleftrightarrow \begin{pmatrix} a_{1i} \\ \vdots \\ a_{hi} \end{pmatrix} \in \underline{X},\ for\ all\ i,\ 1 \leq i \leq h\ ,$$

where $a_i \in E_2^n$ $1 \leq i \leq h$, and $a_i = (a_{i1}, \ldots, a_{in})$. The relation $\underline{X}$ is written as a matrix whose columns are elements of the relation $\underline{X}$. Then the set of functions preserving $\underline{X}$, denoted by $Pol\ \underline{X}$ is defined by:

$$Pol\ \underline{X} = \left\{ f \mid \begin{pmatrix} a_1 \\ \vdots \\ a_h \end{pmatrix} \in \underline{X} \Longrightarrow \begin{pmatrix} f(a_1) \\ \vdots \\ f(a_h) \end{pmatrix} \in \underline{X} \right\}.$$

Intersection of sets $X_1, \ldots, X_k$ will be denoted by $X_1 \ldots X_k$. Let $\overline{X} = H \backslash X$ for each $X \subset H$.

Theorem 1. [8] *There are exactly five P_2 - maximal sets (clones):*

- $T_0 = Pol(0) = \{f | f(0, \ldots, 0) = 0\}$, *the set of functions preserving 0,*
- $T_1 = Pol(1) = \{f | f(1, \ldots, 1) = 1\}$, *the set of functions preserving 1,*
- $S = Pol \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \{f | f(x_1 + 1, \ldots, x_n + 1) = f(x_1, \ldots, x_n) + 1$, *for each* $x_i \in \{0, 1\}$, $1 \leq i \leq n$, *the set of selfdual functions,*
- $L = \{ f \mid f(x_1, \ldots, x_n) = a_0 + a_1 x_1 + \ldots + a_n x_n$, *for some* $a_i \in \{0, 1\}$, $0 \leq i \leq n\}$, *the set of linear functions, and*
- $M = Pol \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, = \{f | x_1 \leq y_1 \wedge \ldots \wedge x_n \leq y_n \Longrightarrow f(x_1, \ldots, x_n) \leq f(y_1, \ldots, y_n)\}$, *the set of monotone functions.*

Classes of functions of bases P_2 are determined in [7].

2. Some modifications of the propositional algebra

A superposition of functions of the system $\{f_1, \ldots, f_s, \ldots\} \subset P_2$ is:

a) any function which can be obtained by the change of variables;

b) any function which can be obtained by change of variables, the equalizing of arguments and addition of dummy (fixed) argument from the function $F(F_1(y_{11}, \ldots, y_{1m_1}), \ldots, F_p(y_{p1}, \ldots, y_{pm_p}))$, where $F(y_1, \ldots, y_p)$ is a superposition of functions of system and $F_i(y_{11}, \ldots, y_{im_i})$ is superposition of functions of system or y_i, $i = 1, \ldots, p$.

If we modify the definition of superposition, then we obtain a modification of the propositional algebra. Some of these modifications are important by reason of their applications. We shall present a certain modifications.

Firstly, we define a modified propositional algebra which uses in consideration of logical structures of computers [1]. Instead of composition $F(F_1(y_{11},\ldots,y_{1m_1}),\ldots,F_p(y_{p1},\ldots,Y_{pm_p}))$ the considered modified propositional algebra uses the composition $\otimes$, defined by

$$(f\otimes g)(x_1,\ldots,x_{m+n-1}) = f(g(x_1,x_2,\ldots,x_n),g(x_2,x_3,\ldots,x_{n+1}),\ldots,\\ g(x_m,x_{m+1},\ldots,x_{m+n-1})),$$

where f and g are given $m-ary$ and $n-ary$ functions respectively.

Algebra defined by operation $\otimes$ is called **Algebra Φ^0**. In [2] Ceitlin has solved the problem of functional completeness for the Algebra Φ^0 algebra, and the most important sets of the Algebra Φ^0 are:

$T_{ij}=\{f|f(0,\ldots,0)=i,\ f(1,\ldots,1)=j\}$, $0\le i,j\le 1$,

$M'=\{f|f\in\Phi^0,\ f'\in M\}$, where $0'=1$ and $1'=0$ and $f'(x_1,\ldots,x_n)=(f(x_1,\ldots,x_n))'$, or ,

$M'=\{f|f^+\in M\}$, i.e. the set of non-incresing or dual monotone functions,

$A=T_{00}\cup T_{01}\cup T_{11}$,

$B=T_{00}\cup T_{11}\cup S$,

$C=T_{00}\cup T_{11}\cup M\cup M'$, and

$D=T_{01}\cup T_{10}\cup\{0,1\}$.

Theorem 2. [2] *There are exactly four Φ^0-maximal sets: A, B, C, and D.*

Gindikin described in [3] some modifications of the propositional algebra and their lattice of closed sets. From the corresponding lattice we can easily determine the maximal sets of any modification.

Definition 1. *We will say that a function f is obtained from the set $F = \{f_1, \ldots, f_s, \ldots\} \subset P_2$ by means of* **extended superposition** *if it may be obtained from F by means of superposition $h(x_1, \ldots, x_{m+n-1}) = f(g(x_1, \ldots, x_n), x_{n+1}, \ldots, x_{m+n-1})$ over the set $F \cup \{0,1\} = \{0, 1, f_1, \ldots, f_s, \ldots\}$. The corresponding algebra is called* **ES-algebra**.

Thus a set of functions $F \subset P_2$ is complete in ES-algebra if and only if $F \cup \{0, 1\}$ is complete with constants in algebra of logic.

Theorem 3. [3] *There are exactly two maximal sets (clones) of the ES-algebra: L and M.*

Definition 2. **GS-algebra** *is an algebra with* **global superposition** *over a set $F \subset P_2$ obtained by replacing a superposition of functions defined by $h(x_1, \ldots, x_{m+n-1}) = f(g(x_1, \ldots, x_n), x_{n+1}, \ldots, x_{m+n-1})$ with the form $f(f_1(y_{11}, \ldots, y_{1m_1}), \ldots, f_p(y_{p1}, \ldots, y_{pm_p}))$, where $f, f_1, \ldots, f_p \in F$.*

In other words, if an argument of a function $f \in F$ is replaced by a function from F, then all arguments of f must be replaced by functions from F.

Theorem 4. [3] *There are exactly 6 maximal sets of the GS-algebra: T_0, T_1, S, L, M and K, where $K = \{f | f(0, \ldots, 0\} = f(1, \ldots, 1)\}$ and the set K is no clone.*

Definition 3. *The dual function of the function f is the function $f^{+}(x_1, \ldots, x_n) = f(x_1 + 1, \ldots x_n + 1) + 1$.*

Definition 4. *The function f is* **selfdual representable** *in terms of the set $F \subset P_2$ if f and f^{+} may be represented by a superposition defined by $h(x_1, \ldots, x_{m+n-1}) = f(g(x_1, \ldots, x_n), x_{n+1}, \ldots, x_{m+n-1})$ of the functions from F. The corresponding algebra is called the* **SP-algebra**.

Theorem 5. [3] *There are exactly 6 maximal sets (clones) of the SP-algebra: L, D, C, S, N_0 and N_1, where*

> $D = \{0, 1\} \cup \{x_{i_1} \vee \ldots \vee x_{i_l} | \{i_1, \ldots, i_l\} \subseteq \{1, \ldots, n\},\ n \geq 1\}$, *the set of all disjunctions,*

$C = \{0,1\} \cup \{x_{i_1} \ldots x_{i_l} | \{i_1, \ldots, i_l\} \subseteq \{1, \ldots, n\},\ n \geq 1\}$, *the set of all conjunctions,*

$N_0 = Pol \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, = \{f | f(\alpha_1, \ldots, \alpha_n) = f(\beta_1, \ldots, \beta_n) = 0 \Longrightarrow$ $\alpha_i = \beta_i = 0$ *for some*; $i,\ 1 \leq i \leq n\}$.

$N_1 = Pol \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, = \{f | f(\alpha_1, \ldots, \alpha_n) = f(\beta_1, \ldots, \beta_n) = 1 \Longrightarrow$ $\alpha_i = \beta_i = 1$ *for some*; $i,\ 1 \leq i \leq n\}$.

3. Symmetric functions

The symmetry of propositional functions usually simplifies the synthesis of switching circuits. Moreover, symmetric functions have algebraic properties which make it desirable to treat them as a separate class. The algebraic treatment of symmetric function is derived from a general definition of symmetric function and a number of theorems, first stated by Shanon [9].

Definition 5. *A function $f(x_1, \ldots, x_n)$ is said to be* **symmetric** *if*

$$f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n),$$

where $(y_1, \ldots, y_n)$ is an arbitrary permutation of $(x_1, \ldots, x_n)$.

Definition 6. S-basis *is a basis which contains only symmetric functions.*

Starting from above definitions the following property of symetric functions can be easily proved ([4], p. 178.):

Lemma 1. *If the perfect disjunctive normal form (***PDNF***) of a n-place symmetric function contains every unit constituent with m, $0 \leq m \leq n$, unnegated variables, then that PDNF contains every unit constituent with the same property.*

Definition 7. *A propositional function is said to be basic symmetric function if and only if every unit constituent of its PDNF can be obtained by some permutation from an arbitrary unit constituent.*

A basic symmetric function is uniquely determined by two numbers - the number of independent variables n and so called **a-number**, i.e. number of unnegated variables k contained in an arbitrary unit constituent of PDNF of that function.

Let S_k^n denote n-place basic symmetric function with a-number k. Given n, there exist exactly $n+1$ basic symmetric functions denoted with $S_0^n, S_1^n, \ldots, S_n^n$.

Lemma 2. [4] *Every symmetric function can be uniquely represented as a disjunction of basic symmetric functions.*

This property enables a suitable way of notation for symmetric functions writing:

$$S_{k_1}^n \vee S_{k_2}^n \vee \ldots S_{k_m}^n \overset{\text{def}}{=} S_{k_1,k_2,\ldots,k_m}^n, \ (n \leq 1).$$

The constants 0 and 1 are symmetric functions and we will denote them by S_0^n and $S_{0,1,\ldots,n}^n$ respectively.

It can be easily proved that the number of n-place symmetric functions is 2^{n+1}.

Let $k/X/_s(n)$ denote the number of n-place symmetric functions of the set or class X, and $k/X/_s(\leq n)$ denote the number of symmetric functions of the set or class X depending on at most n variables.

By N_i^n and $N_i^{(\leq n)}$ we denote the numbers of S-bases consisting from i n-place functions and functions depending on at most n variables respectively. The corresponding numbers of symmetric pivotal incomplete sets we denote by P_i^n and $P_i^{(\leq n)}$.

We obtain the complete classification of symmetric functions of some set if for every class of functions we determine the number of n-ary symmetric functions of this class, and calculate the numbers N_i^n, $N_i^{(\leq n)}$, P_i^n and $P_i^{(\leq n)}$.

The complete classification of symmetric functions of the propositional algebra are presented in [12].

In this paper we present the complete classification of symmetric functions for some modifications of the propositional algebra.

4. Classification of the algebra Φ^0

Theorem 6. [11] *There are exactly nine classes of functions of the algebra Φ^0.*

Classes are presented in the following table.

	1	2	3	4	5	6	7	8	9
A	1	1	1	0	1	0	0	0	0
B	1	1	0	1	0	1	0	0	0
C	1	0	1	1	0	0	1	0	0
D	0	0	0	0	0	0	0	1	0

Theorem 7. [11] *There are exactly eight classes of bases of the algebra Φ^0:*

I - classes of rank 2: (1, 8),

II - classes of rank 3: (7, 2, 8), (4, 2, 8), (3, 2, 8), (6, 3, 8), (4, 3, 8), (5, 4, 8),

III - classes of rank 4: (5, 6, 7, 8).

Theorem 8. [10] *The number of pivotal incomplete sets of the algebra Φ^0 is 27:*

I - eight classes of rank 1: (1) - (8),

II - fifteen classes of rank 2: (2, 3), (2, 4), (2, 7), (2, 8), (3, 4), (3, 6), (3, 8), (4, 5), (4, 8), (5, 6), (5, 7), (5, 8), (6, 7), (6, 8), (7, 8),

III - four classes of rank 3: (5, 6, 7), (5, 6, 8), (5, 7, 8), (6, 7, 8).

The complete classification of symmetric functions of the algebra Φ^0 is given in [10].

5. Classification of the ES-algebra

All four possible classes are nonempty. The class $/L,M/$ contains the function x, 0 and 1. The class $/L/$ contains the function $S_0^1 = 1 + x$, $S_{1,3,5,\dots}^n = x_1 + \dots + x_n$ and $S_{0,2,4,\dots}^n = 1 + x_1 + \dots + x_n$, for $n \leq 2$. The symmetric functions $S_{0,2}^2$ and $S_{k,k+1,\dots,n}^n$ for $0 < k \leq n$ and $n \geq 3$ are of the class $/M/$. The others symmetric functions (for example $S_{0,2,3}^3$) are of the class $/\emptyset/$.

From this we obtain the following theorem:

Theorem 9. *There is only one class of bases of the rank 1: ($/\emptyset/$), and only two classes of pivotal incomplete sets ($/L/$), and ($/M/$) (rank is 1).*

The numbers $k/X/_s(n)$ are presented in Table 1., and numbers $k/X_s(\leq n)$ in Table 2.

X	$n \leq 1$	$n = 2$	$n > 2$
$/L,M/$	$2+n$	2	2
$/L/$	n	2	2
$/M/$	0	1	n
$/\emptyset/$	0	2	$2^{n+1} - n - 4$

Table 1.

X	$n \leq 1$	$n = 2$	$n > 2$
$/L,M_1/$	$2+n$	3	3
$/L/$	n	3	$2n-1$
$/M/$	0	1	$\frac{n(n+1)}{2} - 2$
$/\emptyset/$	0	3	$2^{n+2} - \frac{9n}{2} - \frac{n^2}{2} - 2$

Table 2.

The numbers of S-bases and pivotal incomplete sets of each rank we determine in the following way:

$$N_1^n = k/\emptyset/_s(n),$$

$N_1^{\leq n} = k/\emptyset/_s(\leq n)$,

$N_2^n = k/L/_s(n) \cdot k/M/_s(n)$,

$N_2^{\leq n} = k/M/_s(\leq n)$,

$P_1^n = k/L/_s(n) + k/M/_s(n)$, and

$P_1^{\leq n} = k/L/_s(\leq n) + k/M/_s(\leq n)$.

6. Classification of the GS-algebra

From the definition of the set K we obtain the following lemma:

Lemma 3.

$$f \in T_0\overline{T_1} \cup \overline{T_0}T_1 \Longleftrightarrow f \in K.$$

The first five sets are maximal sets of the propositional algebra also. Hence, the classification of the propositional algebra (Krnić, [7]) can be used. It follows from above Lemma that each class of functions of the propositional algebra corresponds to exactly one class of functions of the GS-algebra. All 15 classes of functions of the GS-algebra we shown in the following table:

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
T_0	1	0	1	1	1	0	0	1	0	0	0	0	0	1	0
T_1	1	1	0	1	0	1	0	0	1	0	0	0	0	1	0
S	0	1	1	0	1	1	1	1	1	1	0	0	0	1	0
L	1	1	1	0	0	0	1	0	0	1	1	1	0	1	0
M	1	1	1	1	1	1	1	0	0	0	1	0	1	1	0
K	1	0	0	1	0	0	1	0	0	1	1	1	1	1	1

Hence we obtain the following two theorems:

Theorem 10. *There are exactly 67 classes of bases of the GS-algebra:*

- *one of the rank 1: (14),*
- *12 classes of the rank 2: (1, 2), (1, 3), (1, 5), (1, 6), (1, 7), (1, 8), (1, 9), (1, 10), (2, 4), (3, 4), (4, 7), (4, 10),*

- *52 classes of the rank 3: (2, 3, 7), (2, 3, 10), (2, 3, 11), (2, 3, 12), (2, 3, 13), (2, 3, 15), (2, 5, 7), (2, 5, 10), (2, 5, 11), (2, 5, 12), (2, 5, 13), (2, 5, 15), (2, 7, 8), (2, 8, 10), (2, 8, 11), (2, 8, 12), (2, 8, 13), (2, 8, 15), (3, 6, 7), (3, 6, 10), (3, 6, 11), (3, 6, 12), (3, 6, 13), (3, 6, 15), (3, 7, 9), (3, 9, 10), (3, 9, 11), (3, 9, 12), (3, 9, 13), (3, 9, 15), (4, 5, 11), (4, 5, 12), (4, 6, 11), (4, 6, 12), (4, 8, 11), (4, 8, 12), (4, 9, 11), (4, 9, 12), (5, 6, 7), (5, 6, 10), (5, 6, 11), (5, 6, 12), (5, 7, 9), (5, 9, 10), (5, 9, 11), (5, 9, 12), (6, 7, 8), (6, 8, 10), (6, 8, 11), (6, 8, 12), (7, 8, 9), (8, 9, 11),*
- *two classes of the rank 4: (8, 9, 10, 13), (8, 9, 12, 13).*

Theorem 11. *There are 82 classes of pivotal incomplete sets of the GS-algebra:*

- *14 classes of the rank 1: (1) - (15) except the class (14),*
- *54 classes of the rank 2: (2, 3), (2, 5), (2, 7), (2, 8), (2, 10), (2, 11), (2, 12), (2, 13), (2, 15), (3, 6), (3, 7), (3, 9), (3, 10), (3, 11), (3, 12), (3, 13), (3, 15), (4, 5), (4, 6), (4, 8), (4, 9), (4, 11), (4, 12), (5, 6), (5, 7), (5, 9), (5, 10), (5, 11), (5, 12), (5, 13), (5, 15), (6, 7), (6, 8), (6, 10), (6, 11), (6, 12), (6, 13), (6, 15), (7, 8), (7, 9), (8, 9), (8, 10), (8, 11), (8, 12), (8, 13), (8, 15), (9, 10), (9, 11), (9, 12), (9, 13), (9, 15), (10, 11), (10, 13), (12, 13),*
- *14 classes of the rank 3: (5, 6, 13), (5, 6, 15), (5, 9, 13), (5, 9, 15), (6, 8, 13), (6, 8, 15), (8, 9, 11), (8, 9, 12), (8, 9, 13), (8, 9, 15), (8, 10, 13), (8, 12, 13), (9, 10, 13), (9, 12, 13).*

The complete classification of symmetric functions of the GS-algebra is obtained in [3], because the classes of the GS-algebra and the classes of the propositional algebra correspond.

7. Classification of the SP-algebra

We obtain the classification of the SP-algebra from the consideration of some cases.

a) $f \in L$

Function 0 is in the set $LDC\overline{S}\overline{N_0}N_1$, and function 1 is in set $LDC\overline{S}N_0\overline{N_1}$. Function x is in the set $LDCSN_0N_1$. The set $L\overline{DC}S\overline{N_0N_1}$ contains the functions $1+x$, $a_0+x_1+\ldots+x_{2m+1}$, and the set $L\overline{DCSN_0N_1}$ contains the functions $a_0+x_1+\ldots+x_{2m}$.

b) $f \in \overline{L}D$

The set $\overline{L}D$ contains the functions $S^n_{1,2,\ldots,n} = x_1 \vee \ldots \vee x_n$ for $n \geq 2$. It follows from $f(0,1,x_3,\ldots,x_n) = f(1,0,x_3+1,\ldots,x_n+1) = 1$ that $f \in \overline{SN_1C}$. If $f(x_1,\ldots,x_n) = 0$, then $x_1 = \ldots = x_n = 0$, therefore we have $f \in N_0$.

c) $f \in \overline{LDC}$

The set $\overline{LDC}$ consists of functions $S^n_n = x_1 \ldots x_n$, $(n \geq 2)$, which are in the set $\overline{SN_0}N_1$ (proof is analogous).

d) $f \in \overline{LDCS}$

Lemma 4. $f \in SN_0 \Longrightarrow f \in N_1$, $f \in SN_1 \Longrightarrow f \in N_0$

Proof. We shall only prove the first assertion, because the proof of the second assertion is analogous. Let $f(\alpha_1,\ldots,\alpha_n) = f(\beta_1,\ldots,\beta_n) = 1$. Since $f \in S$, we have $f(\alpha_1+1,\ldots,\alpha_n+1) = f(\beta_1+1,\ldots,\beta_n+1) = 0$ It follows from $f \in N_0$ and that there exist i such that $\alpha_i+1 = \beta_i+1 = 0$, therefore, we obtain $\alpha_i = \beta_i = 1$. Hence, $f \in N_1$. □

The functions $S^{2m+1}_{m+1,m+2,\ldots,2m+1}$ are in the set N_0N_1, and the functions $S^{2m+1}_{k_0,\ldots,k_m}$ for $k_i \in \{i, 2m+1-i\}$, $1 \leq i \leq m$ (except the function $S^{2m+1}_{m+1,\ldots,2m+1}$) are in the set $\overline{N_0N_1}$.

e) $f \in \overline{LDCS}$

Lemma 5. $f \in N_0N_1 \Longrightarrow f \in S$.

Proof. Suppose that $f \in \overline{S}$, then there is a vector α such that $f(\alpha_1,\ldots,\alpha_n) = f(\alpha_1+1,\ldots,\alpha_n+1) = c$. If $c=0$ then $f \notin N_0$, and if $c=1$ then $f \notin N_1$. □

The set $\overline{N_0}N_1$ contains the symmetric functions $S^n_{k_1,\ldots,k_m}$, where $0 \leq k_1 < k_2 < \ldots < k_m \leq n$ and $2k_1 > n$ (for example $S^5_{4,5}$).

Let k be the minimal number in the set $\{0,1,\ldots,n\}$ which is not in the set $\{k_1,\ldots,k_m\}$. The set $N_0\overline{N_1}$ contains the symetric functions $S^n_{k_1,\ldots,k_m}$, where $2k < n$ (for example $S^5_{2,3,4,5}$).

All the others symmetric functions are in the set $\overline{N_0 N_1}$ (for example $S^3_{1,2}$).

From above definitions we obtain the following theorems:

Theorem 12. *There are 12 classes of functions of the SP-algebra.*

	1	2	3	4	5	6	7	8	9	10	11	12
L	0	0	0	0	1	1	1	1	1	1	1	0
D	0	0	1	1	0	1	1	1	1	1	1	0
C	0	0	1	1	1	0	1	1	1	1	1	0
S	1	1	0	1	1	1	0	0	1	1	1	0
N_0	1	0	1	1	0	1	0	1	0	1	1	0
N_1	0	1	1	1	1	0	0	1	1	0	1	0

Theorem 13. *There are exactly 27 classes of bases of the SP-algebra:*

- *one classes of the rank 1: (11),*
- *22 classes of the rank 2: (1, 8), (1, 9), (2, 8), (2, 10), (3, 5), (3, 6), (3, 9), (3, 10), (4, 5), (4, 6), (4, 7), (4, 8), (4, 9), (4, 10), (5, 6), (5, 8), (5, 10), (6, 8), (6, 9), (8, 9), (8, 10), (9, 10),*
- *5 classes of the rank 3: (1, 2, 7), (1, 3, 7), (1, 5, 7), (2, 3, 7), (2, 6, 7),*

Theorem 14. *There are 20 classes of the pivotal incomplete sets:*

- *10 classes of the rank 1: (1) - (10),*
- *10 classes of the rank 2: (1, 2), (1, 3), (1, 5), (1, 7), (2, 3), (2, 6), (2, 7), (3, 7), (5, 7), (6, 7).*

The numbers $k/X/_s(n)$ are given in the following table:

	$n \leq 1$	$n = 2k > 1$	$n = 2k+1 > 1$
1, 2	1	1	1
3	n	0	2
4	0	2	0
5, 6	0	1	1
7	0	0	1
8	0	0	$2^{\frac{n+2}{2}} - 1$
9, 10	0	$2^{\frac{n}{2}} - 2$	$2^{\frac{n+1}{2}} - 2$
11	0	$2^{n+1} - 2^{\frac{n+2}{2}} - 2$	$2^{n+1} - 3 \cdot 2^{\frac{n+1}{2}} - 2$
12	n	0	0

The numbers $k/X/_s(\leq n)$ are given in the following table:

	$n \leq 1$	$n > 1$
1, 2	1	1
3	n	$n - \frac{1+(-1)^n}{2}$
4	0	$n - \frac{1+(-1)^{n+1}}{2}$
5, 6	0	$n - 1$
7	0	$\left[\frac{n-1}{2}\right]$
8	0	$2^{\left[\frac{n+3}{2}\right]} - \left[\frac{n-1}{2}\right] - 4$
9, 10	0	$2^{\left[\frac{n+1}{2}\right]} \cdot \frac{3+(1+(-1)^n)}{2} - 2n - 4$
11	0	$2^{n+2} - (9 + (-1)^n) \cdot 2^{\left[\frac{n+1}{2}\right]} - 2n - 10$
12	n	1

From above theorems and data we can easily determine the numbers N_i^n, $N_i^{(\leq n)}$, P_i^n and $P_i^{(\leq n)}$.

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REZIME

KLASIFIKACIJA NEKIH MODIFIKACIJA ISKAZNE ALGEBRE

U radu je izvršena klasifikacija funkcija nekih modifikacija iskazne algebre, kao i potpuna klasifikacija odgovarajućih simetričnih funkcija.

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