Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 23, 2 (1993), 289 - 303

Review of Research
Faculty of Science
Mathematics Series

ON RANK 2 GREEDOIDS

Dragan M. Acketa
Institute of Mathematics, University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

Rank 2 greedoids are represented by some graphs with marked vertices. This representation is used for producing the list of all non-isomorphic rank 2 greedoids on at most 7 elements. The work is partly computer aided.

AMS Mathematics Subject Classification (1991): 05B35, 05C30
Key words and phrases: greedoids, rank 2, simple graphs with marked vertices, enumeration.

1. Preliminaries

An n-set is a set of cardinality n. Sets are denoted without brackets and commas.

A *greedoid* ([5]) on a finite set (*ground-set*) E is an ordered pair (E, F), where F is a family of so-called *feasible* subsets of E, which satisfies the following conditions:

(1) The empty set is feasible.

(2) Each non-empty feasible set contains an element, the removal of which leaves a feasible subset.

(3) Given two feasible sets X and Y of cardinalities k and $k+1$ respectively, there exists an element in Y, which can be added to X so that another feasible $(k+1)$-set is obtained.

Two greedoids are *isomorphic* if there is a bijection between their ground-sets, which preserves the feasible sets. The *rank* of a greedoid is the maximal cardinality of a feasible set. A greedoid is *full* iff each element of the ground-set belongs to a feasible set.

2. Introduction

All non-isomorphic greedoids on at most 5 elements were constructed in [2] with computer aid. A complete search, which included a brute-force "erection technique", was applied. In other words, given a rank k greedoid (represented by feasible sets of cardinality at most k), all possibilities for addition of (a family of) feasible $(k+1)$-sets were examined (and only non-isomorphic possibilities satisfying the greedoid axioms were kept in the generated list).

Rank 2 is the smallest rank, which gives non-trivial greedoids (the empty greedoid is the only rank 0 greedoid and it is obvious that there exist n non-isomorphic rank 1 greedoids on an n-set). A formula for the number of non-isomorphic rank 2 matroids (which constitute an important subclass of rank 2 greedoids) was obtained in [1]. Note, however, that, generally speaking, matroids are very special greedoids (e. g., there are 25612 non-isomorphic greedoids on 5 elements (of all ranks), only 38 of which are matroids). A similar conclusion remains valid even for the restriction to the rank 2 case (see the table at the end of Section 5 below).

In this paper we generate all the non-isomorphic rank 2 greedoids on at most 7 elements (comparing with the previous results, we have the list of rank 2 greedoids on 6 and 7 elements as a new result). Our technique uses auxiliary simple graphs with marked vertices. We adjoin the vertices, the marked vertices and the edges of these graphs to the elements of the ground-set, to the feasible 1-sets and to the feasible 2-sets respectively.

3. Construction

The main idea of our procedure is to reverse the erection approach in the rank 2 case (to "build" the greedoids starting from the "roof") and to use the lists of non-isomorphic greedoids, as well as our intuition on graphs. The work is divided into three stages:

(a) Producing the list of all non-isomorphic simple graphs on at most 7 vertices. We can rely upon the figures given in the Appendix of [4] (the number of all non-isomorphic simple graphs (in particular, the number of connected such graphs) on p vertices and q edges (for small p and q) was obtained there by the use of the Polya's theorem).

(b) Producing all the combinations of vertices (= all subsets of the ground-set), which correspond to the regularly marked vertices. Each such combination corresponds to a rank 2 greeedoid, which belongs to a class of rank 2 greedoids with the fixed family of feasible 2-sets. This class of greedoids is associated with the graph itself.

(c) Elimination of superfluous greedoids (that is, isomorphic copies) arising in stage (b).

A computer was used for performing stage (b), as well as for simplifying stage (a). Namely, we need only simple graphs with at most half of the maximal possible number of edges for input. The remaining simple graphs can be obtained by complementation:

Algorithm

REPEAT

  READ a simple graph G with at most $0.25 \cdot n \cdot (n-1)$ edges
  (* Graphs are represented by the list of their edges. Their vertices are denoted by 1,2, ...,n *)

 1: FOR each non-empty subset S of $\{1, 2, ..., n\}$ DO
   (* Test (2) *)
   IF each edge of G has at least one of its vertices in S
   THEN (* Test (3) *)
     IF for each vertex v in S, each edge of G has at least one of its vertices in $A(v, G)$
     (* where $A(v, G)$ denotes the auxiliary set consisting of v and all the vertices, which are incident to v in G *)

THEN S corresponds to a rank 2 greedoid. Keep S as a candidate for the list of non-isomorphic rank 2 greedoids.
IF G has less than $0.25 \cdot n \cdot (n-1)$ edges
THEN replace the graph G by its complementary graph $c(G)$ and goto 1.
UNTIL the input list of simple graphs on n vertices is exhausted.

Remark. This is a modification of the general "greedoidity" testing algorithm, adjusted for the rank 2 case and equipped with the use of complementation. Testing (2) and (3) requires two, respectively three nested loops in the general case. Here we save the inner loop by asking "if an edge has at least one of its two vertices in a given set..."

When the "non-computer" part of our work is considered, by far the most elaborate task is the producing of the lists of simple graphs on 7 vertices (the number of such graphs can be found in [3] or [4], but we have had no access to a catalogue of such graphs). The lists of all non-isomorphic graphs on at most six vertices can be found in [3].

The stage (c) of our construction is performed by hand very quickly. Given a "greedoidic" graph (i.e., a simple graph which has at least one associated rank 2 greedoid), the computer produces the list of "greedoidic" subsets of the vertex-set (each such subset corresponds to the family of feasible 1-sets of an associated greedoid).

First of all, we can be sure that there are no mutually isomorphic associated greedoids in the frequent cases when two "greedoidic" subsets of the same cardinality do no exist. In other cases we have to decide which of the equicardinal "greedoidic" subsets have mutually isomorphic positions w.r.t. the graph (more precisely, which of them can be mapped to each other by a graph automorphism). A quick way to try doing this is to primarily determine (using intuition and some easy graph invariants) the orbits of the automorphism group of the vertex-set. We have to compare afterwards whether some two of the considered subsets are related to these orbits in the same way (in most cases it suffices to check whether each orbit has the same intersection cardinality with both sets).

A nice feature of graph complementation is that it preserves the orbits and all mutually (non-) automorphic subsets of vertices. This is a consequence of the fact that the automorphism groups $\mathrm{Aut}(G)$ and $\mathrm{Aut}(c(G))$ coincide (the mutual relative positions of vertices are preserved after the presence and the absence of edges are interchanged). It follows that we can use our intuition about a graph G in order to treat the complementary graph $c(G)$ with a larger number of edges.

Example.

Let the simple graph G on vertices denoted by 1, 2, ..., 7, be given, the edges of which are 12, 13, 14, 15, 16, 23. The associated (non-full) greedoids are determined by the "greedoidic" subsets 12, 13, 123 respectively. The "greedoidic" subsets corresponding to the (full) greedoids associated to the complementary graph $c(G)$ are 4567, 23457, 23467, 23567, 24567, 34567, 234567 respectively. It is obvious that the orbits of $\mathrm{Aut}(G)$ on the vertex-set are $\{1\}, \{2,3\}, \{4,5,6\}$ and $\{7\}$. It follows that the "greedoidic" subsets 12 and 13, respectively 23457, 23467 and 23567, respectively 24567 and 34567, – are mutually automorphic (consider the intersection cardinalities with the orbits!). Thus in the list of non-isomorphic rank 2 greedoids we keep solely the greedoids determined by the "greedoidic" subsets 12 and 123 (associated to G) and determined by 4567, 23457, 24567 and 234567 (associated to $c(G)$).

In some cases the orbits on the vertex-set are not sufficient. Thus all four vertices of a 4-cycle are in the same orbit, but only two of the six 2-subsets of the vertex-set are "greedoidic" (the corresponding two rank 2 greedoids are mutually isomorphic).

4. Denotations of non-isomorphic rank 2 greedoids in the list

We use denotations of the form GRAPH: GREEDOIDS, where
GRAPH = denotation of a simple graph on n vertices ($2 \leq n \leq 7$) and k edges $0 \leq k \leq (n \cdot (n-1))/4$)
GREEDOIDS = denotation of the list of all non-isomorphic greedoids, which are associated either to the denoted graph or to the complementary graph.

The graph vertices are elements of the set $\{1, \ldots, n\}$. GRAPH is replaced by the corresponding edge-set. Each edge (= feasible 2-set) is written as a 2-set of vertices. Edge denotations are mutually separated by commas. The empty edge-set is denoted by 0. The number of listed edges is never greater than $(n \cdot (n-1))/4$, (the larger numbers of edges can be treated by complementation).

GREEDOIDS are denoted by the families of feasible 1-sets (= subsets of marked vertices). The denotation of each greedoid in the list is preceded by a short line "-". The greedoids which are denoted in the list after the letter "C" are related to the complementary graph (their list of feasible 2-sets is complementary to the given one.)

The sublists of the form GRAPH: GREEDOIDS are succesively written in our list, following each other with little space in between. No effort is made to keep such sublists within lines of the list (sublists can be cut at any point by end-of-line).

The feasible empty set is always preassumed.

Example.

The sublist 12, 13: -1-12-23-123 C-245-2345 replaces the following six rank 2 greedoids on 5 elements (listed by all their feasible sets):

∅, 1, 12, 13

∅, 1, 2, 12, 13

∅, 2, 3, 12, 13

∅, 1, 2, 3, 12, 13

∅, 2, 4, 5, 14, 15, 23, 24, 25, 34, 35, 45

∅, 2, 3, 4, 5, 14, 15, 23, 24, 25, 34, 35, 45

5. List of all non-isomorphic rank 2 greedoids on at most 7 elements

2 elements: ∅: C-1-12
3 elements: ∅: C-12-123 12: -1-12 C-3-12-13-123
4 elements: ∅: C-123-1234 12: -1-12 C-34-123-134-1234 12, 13: -1-12-23-123 C-24-234 12, 34: C-12-123-1234 12, 13, 23: -12-123 12, 13, 24: -12 12, 13, 14: -1-12-123-234-1234
5 elements: ∅: C-1234-12345 12: -1-12 C-345-1234-1345-12345 12, 13: -1-12-23-123 C-245-2345 12, 34: C-125-1234-1235-12345 12, 13, 23: -12-123 C-45-145-1234-1245-12345 12, 13, 24: -12 C-345 12, 13, 14: -1-12-123-

234-1234 C-235-2345 12, 13, 45: -245-2345 12, 13, 23, 45: C-45-123-145-1234-1245-12345 12, 13, 24, 34: -14-123-1234 12, 13, 14, 23: -12-123 C-45-235-245-2345 12, 13, 14, 25: -12 C-345 12, 13, 14, 15: -1-12-123-1234-2345-12345 C-234-2345 12, 13, 14, 23, 24: -12-123 -134-1234 12, 13, 14, 25, 35: -123 12, 13, 14, 15, 23: -12-123 12, 13, 23, 24, 35: -23-123
6 elements: $\emptyset$: C-12345-123456 12: -1-12 C-3456-12345-13456-123456 12, 34: C-1256-12345-12356-123456 12, 13: -1-12-23-123 C-2456 -23456 12, 13, 23: -12-123 C-456-1456-12345-12456-123456 12, 13, 24: -12 C-3456 12, 13, 14: -1-12-123-234-1234 C-2456-23456 12, 34, 56: C-1234-12345-123456 12, 13, 45: C-2456-23456 12, 13, 14, 15: -1-12-123 -1234-2345-12345 C-2346-23456 12, 13, 14, 25: -12 C-3456 12, 13, 14, 56: C-2356-23456 12, 13, 24, 56: C-3456 12, 13, 24, 34: -14-123-1234 12, 13, 14, 23: -12-123 C-456-2356-2456-23456 12, 13, 23, 45: C-456 -1236-1456-12345-12346-12456-123456 12, 13, 14, 25, 35: -123 12, 13, 14, 23, 45: C-2356 12, 13, 14, 23, 25: -12-123 C-456-3456 12, 13, 14, 15, 23: -12-123 C-456-2346-2456-23456 12, 13, 14, 23, 56: C-456-2356-2456 -23456 12, 13, 23, 45, 46: C-1235-12356 12, 13, 14, 23, 24: -12-123-134 -1234 C-356-3456 12, 13, 14, 15, 26: -12 C-3456 12, 13, 14, 15, 16: -1-12 -123-1234-12345-23456-123456 C-2345-23456 12, 13, 14, 25, 26: -12 C-3456 12, 13, 14, 23, 24, 34: -123-1234 C-56-156-1256-12345-12356 -123456 12, 13, 14, 15, 23, 24: -12-123-134-1234 C-356-3456 12, 13, 14, 23, 25, 35: -123 C-456 13, 14, 15, 23, 24, 25: -12-123-345-1234-1345-12345 12, 13, 14, 23, 24, 56: C-356-3456 12, 13, 23, 45, 46, 56: C-123-1234-12345-123456 12, 13, 14, 15, 23, 45: C-236-2346-23456 12, 13, 14, 15, 26, 36: -123 12, 13, 14, 15, 16, 23: -12-123 C-456-2345-2456-23456 12, 13, 14, 15, 23, 26: -12-123 C-456-3456 12, 13, 14, 25, 26, 34: C-3456 12, 13, 14, 23, 25, 36: -123 C-456 12, 13, 14, 15, 26, 34: C-3456 12, 13, 14, 15, 23, 24, 34: -123-1234 C-56-256-2346-2356-23456 12, 13, 14, 23, 24, 34, 56: C-56-156-1234-1256-12345-12356-123456 12, 13, 14, 15, 23, 24, 35: -123 C-456 12, 13, 14, 23, 24, 35, 45: -134-1234 12, 13, 14, 15, 23, 24, 25: -12-123-1234-1345-12345 C-346-3456 12, 13, 14, 15, 16, 23, 24: -12-123-134-1234 C-356-3456 12, 13, 14, 15, 23, 26, 36: -123 C-456 12, 13, 14, 15, 23, 24, 36: -123 C-456 12, 13, 14, 15, 23, 24, 26: -12-123-1234 C-356-3456 12, 13, 14, 15, 16, 24, 35: C-246-2345-2346-23456 12, 13, 14, 15, 24, 26, 35: C-356-3456 12, 13, 14, 25, 26, 34, 56: C-3456 13, 14, 15, 16, 23, 24, 25: -1345 12, 14, 15, 23, 26, 34, 35: -123

7 elements: ∅: C-123456-1234567 12: -1-12 C-34567-123456-134567-1234567 12, 13: -1-12-23-123 C-24567-234567 12, 34: C-12567-123456-123567-1234567 12, 13, 23: -12-123 C-4567-14567-123456-124567-1234567 12, 13, 14: -1-12-123-234-1234 C-23567-234567 12, 13, 24: -12 C-34567 12, 13, 45: C-24567-234567 12, 34, 56: C-12347-123456 -123457-1234567 12, 13, 14, 25: -12 C-34567 12, 13, 14, 15: -1-12-123-1234-2345-12345 C-23467-234567 12, 13, 14, 56: C-23567-234567 12, 13, 24, 56: C-34567 12, 13, 45, 67: C-24567-234567 12, 13, 24, 34: -14-123-1234 12, 13, 14, 23: -12-123 C-4567-23567-24567-234567 12, 13, 23, 45: C-4567-12367-14567-123456-123467-124567-1234567 12, 13, 14, 25, 26: -12 C-34567 12, 13, 14, 15, 26: -12 C-34567 12, 13, 14, 15, 16: -1-12-123-1234-12345-23456-123456 C-23457-234567 12, 13, 14, 15, 67: C-23467-234567 12, 13, 14, 25, 35: -123 12, 13, 14, 23, 45: C-23567 12, 13, 14, 25, 67: C-34567 12, 13, 14, 15, 23: -12-123 C-4567-23467-24567-234567 12, 13, 14, 23, 25: -12-123 C-4567-34567 12, 13, 23, 45, 46: C-12357-123567 12, 13, 23, 45, 67: C-4567-12345-14567-123456-124567-1234567 12, 13, 14, 23, 24: -12-123-134-1234 C-3567-34567 12, 13, 14, 23, 56: C-4567-23567-24567-234567 12, 13, 14, 25, 26, 27: -12 C-34567 12, 13, 14, 15, 16, 27: -12 C-34567 12, 13, 14, 15, 16, 17: -1-12-123-1234-12345-123456-234567-1234567 C-23456-234567 12, 13, 14, 15, 26, 36: -123 12, 13, 14, 25, 26, 34: C-34567 12, 13, 14, 15, 23, 46: C-23567 12, 13, 14, 15, 16, 23: -12-123 C-4567-23457-24567-234567 12, 13, 14, 15, 23, 26: -12-123 C-4567-34567 12, 13, 14, 23, 25, 36: -123 C-4567 12, 13, 14, 15, 23, 24: -12-123-134-1234 C-3567-34567 12, 13, 14, 23, 25, 35: -123 C-4567 12, 13, 14, 23, 24, 56: C-3567-34567 12, 13, 14, 15, 23, 45: C-2367-23467-234567 13, 14, 15, 23, 24, 25: -12-123-345-1234-1345-12345 12, 13, 23, 45, 46, 56: C-1237-12347-14567-123456-123457-1234567 12, 13, 14, 23, 24, 34: -123-1234 C-567-1567-12567-123456-123567-1234567 12, 13, 14, 23, 45, 67: C-23567 12, 13, 14, 15, 23, 67: C-4567-23467-24567-234567 12, 13, 14, 23, 25, 67: C-4567-34567 12, 13, 23, 45, 46, 57: C-12367 12, 13, 23, 45, 46, 47: C-12356-123567 12, 13, 14, 15, 16, 27, 37: -123 12, 13, 14, 15, 26, 27, 67: C-34567 12, 13, 14, 15, 16, 17, 23: -12-123 C-4567-23456-24567-234567 12, 13, 14, 15, 23, 46, 47: C-23567 12, 13, 14, 15, 16, 23, 47: C-23567 12, 13, 14, 15, 16, 23, 27: -12-123 C-4567-34567 12, 13, 14, 15, 23, 26, 27: -12-123 C-4567-34567 12, 13, 14, 15, 23, 26, 37: -123 C-4567 12, 13, 14, 15, 16, 23, 24: -12-123-134-1234 C-3567-34567 12, 13, 14, 15, 23, 26, 36: -123 C-4567 12, 13, 14, 15, 23, 24, 26: -12-123-1234 C-3567-34567 12, 13, 14, 15, 23, 24, 36: -123 C-4567 12, 13, 14, 23, 25, 35, 67: C-4567 12, 13, 14, 15, 23, 24, 67: C-3567-34567 12, 13, 14, 15, 26, 36, 46: -1234 12, 13, 14, 25, 26, 35, 36: -123 12, 13, 14, 15, 23, 26, 45: -4567-34567 12,

13, 14, 15, 16, 23, 45: C-2367-23457-23467-234567 12, 13, 14, 15, 23, 45, 67: C-2367-23467-234567 12, 13, 14, 25, 26, 34, 56: C-34567 12, 13, 14, 15, 23, 24, 34: -123-1234 C-567-2567-23467-23567-234567 12, 13, 14, 23, 24, 34, 56: C-567-1567-12347-12567-123456-123457-123567-1234567 12, 13, 14, 15, 23, 34, 45: -134 C-2567 12, 13, 14, 23, 24, 35, 45: -134-1234 12, 13, 14, 15, 23, 24, 25: -12-123-1234-1345-12345 C-3467-34567 12, 13, 14, 23, 56, 57, 67: C-4567-23567-24567-234567 12, 13, 14, 15, 16, 17, 23, 24: -12-123-134-1234 C-3567-34567 12, 13, 14, 15, 16, 23, 27, 37: -123 C-4567 12, 13, 14, 15, 16, 23, 24, 37: -123 C-4567 12, 13, 14, 15, 16, 23, 24, 27: -12-123-1234 C-3567-34567 12, 13, 14, 15, 23, 26, 27, 37: -123 C-4567 12, 13, 14, 15, 23, 24, 26, 37: -123 C-4567 13, 14, 15, 16, 17, 23, 24, 25: -1345 12, 13, 14, 15, 26, 27, 36, 37: -123 12, 13, 14, 15, 23, 26, 27, 45: C-4567-34567 12, 13, 14, 15, 16, 23, 45, 67: C-23457 12, 13, 14, 15, 16, 17, 23, 45: C-2367-23456-23467-234567 12, 13, 14, 15, 23, 26, 37, 45: C-4567 12, 13, 14, 15, 16, 23, 27, 45: C-4567-34567 12, 13, 14, 15, 26, 27, 34, 67: C-34567 12, 13, 14, 15, 23, 24, 34, 56: C-23467 12, 13, 14, 15, 23, 24, 26, 34: -123-1234 C-567-3567-34567 12, 13, 14, 15, 16, 23, 24, 34: -123-1234 C-567-2567-23457-23567-234567 12, 13, 14, 23, 24, 56, 57, 67: -3567-34567 12, 13, 14, 15, 16, 23, 24, 56: C-3567-34567 12, 13, 14, 15, 23, 26, 36, 45: C-4567 12, 13, 14, 15, 23, 24, 34, 67: C-567-2567-23467-23567-234567 12, 13, 14, 23, 24, 34, 56, 57: C-12346-123467 12, 13, 14, 15, 23, 24, 26, 36: -123 C-4567 12, 13, 14, 15, 23, 24, 35, 67: C-4567 12, 13, 14, 15, 16, 23, 34, 45: -134 C-2567 12, 13, 14, 15, 23, 24, 36, 46: -134-1234 12, 13, 14, 15, 16, 23, 24, 25: -12-123-1234-1345-12345 C-3467-34567 12, 13, 14, 15, 23, 24, 25, 36: -123 C-4567 12, 13, 14, 15, 23, 24, 25, 67: C-3467-34567 12, 13, 14, 15, 23, 25, 34, 45: -124-1234-2345-12345 12, 13, 14, 23, 24, 34, 35, 45: -134-1234 C-567-1267-1567-12567 13, 14, 15, 23, 24, 25, 36, 46: -1234 13, 14, 15, 16, 23, 24, 25, 26: -12-123-1234-3456-12345-13456-123456 12, 13, 14, 15, 16, 23, 24, 34, 57: C-23467 12, 13, 14, 15, 23, 24, 34, 56, 57: C-23467 12, 13, 14, 15, 16, 17, 23, 24, 34: -123-1234 C-567-2567-23456-23567-234567 12, 13, 14, 15, 16, 23, 24, 27, 34: -123-1234 C-567-3567-34567 12, 13, 14, 15, 23, 24, 26, 34, 37: -123-1234 C-567-4567 12, 13, 14, 15, 23, 26, 27, 36, 45: C-4567 12, 13, 14, 15, 16, 23, 27, 37, 45: C-4567 12, 13, 14, 15, 16, 23, 24, 27, 56: C-3567-34567 12, 13, 14, 15, 16, 23, 27, 34, 56: C-4567 12, 13, 14, 15, 16, 17, 23, 24, 56: C-3567-34567 12, 13, 14, 15, 16, 23, 24, 27, 37: -123 C-4567 12, 13, 14, 15, 16, 17, 23, 24, 35: -123-4567 12, 13, 14, 15, 16, 23, 24, 27, 35: -123 C-4567 12, 13, 14, 15, 23, 24, 26, 35, 37: -123 C-4567 12, 13, 14, 15, 16, 17, 23, 24, 25: -12-123-1234-1345-12345 C-3467-34567 12, 13, 14, 15, 23, 24, 25, 36, 37: -123 C-4567 12, 13, 14, 15, 16, 23, 24, 25, 37: -123 C-4567 12, 13, 14, 15, 16, 23, 24, 25, 27: -12-123-1234-12345 C-3467-34567 12, 13, 14, 15, 16, 23, 27, 34, 47: -124-1234 12, 13, 14, 15, 23, 24, 26, 37, 47: -1234 12, 13, 14, 15, 23, 26, 27, 45, 67: C-4567-34567 12, 13, 14, 15, 16, 17, 23, 45, 67: C-2345-23456-234567 13, 14, 15, 16, 17, 23, 24, 25, 26: -13456 12, 13, 14, 25, 26, 27, 35, 36, 37: -123 12, 13, 14, 23, 24, 34, 56, 57, 67: C-567-1234-1567-12345-12567-123456-123567-1234567 12, 13, 14, 15, 23, 24, 25, 34, 35: -123-1234-1245-12345 C-467-4567 12, 13, 14, 15, 23, 24, 25, 34, 67: C-567-3467-3567-34567 12, 13, 16, 23, 24, 25, 34, 35, 45: C-4567 12, 14, 15, 16, 24, 25, 26, 34, 35: -1245 12, 14, 15, 16, 23, 24, 25, 34, 35: -1245 12, 13, 14, 16, 23, 24, 25, 34, 35: -123-1234 C-567-4567 12, 13, 15, 16, 23, 24, 26, 34, 35: -123 C-4567 12, 13, 14, 15, 16, 23, 24, 34, 56: C-567-2347-2567-23457-23567-234567 12, 13, 14, 15, 16, 24, 25, 34, 35: -123-1234-12345 12, 13, 14, 15, 16, 23, 24, 25, 45: -124-1245 C-367-3467-4567-34567 13, 14, 15, 16, 23, 24, 25, 26, 34: -123-1234 12, 13, 14, 15, 16, 23, 26, 34, 35: -123 C-4567 14, 15, 16, 24, 25, 26, 34, 35, 36: -123-1234-12345-123456 12, 13, 14, 15, 16, 23, 24, 25, 26: -12-123-1234-12345-13456-123456 C-3457-34567 12, 13, 14, 15, 16, 17, 23, 24, 25, 67: C-3467-34567 12, 13, 14, 15, 16, 17, 23, 24, 25, 26: -12-123-1234-12345-13456-123456 C-3457-34567 13, 14, 15, 16, 17, 23, 24, 25, 26, 27: -12-123-1234-12345-123456-134567-1234567 12, 13, 14, 15, 16, 23, 24, 27, 35, 37: -123 C-4567 12, 13, 14, 15, 23, 24, 26, 27, 36, 37: -123 C-4567 12, 13, 14, 15, 16, 17, 23, 24, 25, 36: -123 C-4567 12, 13, 14, 15, 16, 23, 24, 35, 36, 37: -123 C-4567 12, 13, 14, 15, 16, 23, 24, 25, 26, 37: -123 C-4567 12, 13, 14, 15, 16, 23, 24, 27, 34, 37: -123-1234 C-567-4567 12, 13, 14, 15, 16, 17, 23, 24, 25, 34: -123-1234 C-567-3467-3567-34567 12, 13, 14, 15, 16, 23, 24, 27, 34, 35: -123-1234 C-567-4567 12, 13, 14, 15, 23, 24, 26, 34, 37, 45: -1234 C-567 12, 13, 14, 15, 16, 23, 24, 25, 27, 34: -123-1234 C-567-3467-3567-34567 12, 13, 14, 15, 16, 23, 24, 27, 34, 56: C-567-3567-34567 12, 13, 14, 15, 16, 17, 23, 24, 34, 56: C-567-2347-2567-23456-23457-23567-234567 12, 13, 14, 15, 16, 23, 24, 34, 56, 57: C-2347-23467 12, 13, 14, 15, 23, 24, 34, 56, 57, 67: C-23467 12, 13, 14, 15, 16, 23, 27, 34, 37, 47: -1234 12, 13, 14, 15, 16, 17, 23, 24, 35, 45: -125-1234-12345 12, 13, 14, 15, 16, 23, 25, 27, 34, 45: -1235 12, 13, 14, 15, 23, 24, 25, 34, 56, 57: C-3467 12, 13, 14, 15, 16, 23, 24, 25, 37, 45: C-4567 12, 13, 14, 15, 26, 27, 36, 37, 56, 57: -1235 12, 13, 14, 15, 16, 23, 24, 25, 37, 47: -1234 12, 13, 14, 15, 24, 26, 27, 34, 36, 37: -123-1234 12, 13, 14, 15, 23, 24, 25, 34, 35, 45: -1234-12345 C-67-167-1267-12367-123456-123467-1234567 12, 13, 14, 15, 23, 24, 25, 34, 35, 67: C-467-4567 12, 13, 14, 16, 23, 24, 25, 34, 35, 45: -1234 C-567 12, 13, 14, 15, 16, 23, 24, 25, 26, 34: -123-1234 C-567-3457-3567-34567 12, 13, 14, 16, 23, 24, 26, 34, 35, 45: -1234 C-567 12, 13, 14, 15, 16, 23, 24, 25, 36, 45: C-4567 12, 13, 14, 15, 16, 24, 25, 26, 34, 35: -1245 12, 13, 14, 15, 16, 23, 24, 25, 34, 35: -123-1234-1245-12345 C-467-4567 12, 13, 14, 15, 16, 23, 24, 26, 34, 35: -123-1234 C-567-4567 13, 14, 15, 16, 23, 24, 25, 26, 34, 35: -123-1234-1245-12345 12, 14, 15, 16, 24, 25, 26, 34, 35, 36: -1456-12456

The following table summarizes the numerical data for rank 2 greedoids on ≤ 7 elements and their graphic representations. The rows for $GG(n)$ and $GD(n)$ are derived from the list above, while the row for $GC(n)$ is also contained in the computer output:

n	2	3	4	5	6	7
$SG(n)$	2	4	11	34	156	1044
$GG(n)$	1	3	9	25	80	265
$GD(n)$	2	8	25	70	200	592
$GC(n)$	3	12	43	126	380	1114
$MD(n)$	1	3	7	13	23	37

$SG(n) =$ the total number of non-isomorphic simple graphs on n vertices (can be found, e. g., in [3] or [4])

$GG(n) =$ the number of non isomorphic "greedoidic" graphs on n vertices

$GD(n) =$ the number of non-isomorphic rank 2 greedoids on n elements

$GC(n) =$ the total number of greedoids on n elements, which are associated to all non-isomorphic "greedoidic" graphs on n vertices. These greedoids can be considered to be candidates for the list of non-isomorphic greedoids (in all cases we have actually chosen lexicographically the first candidate to remain in the list)

$MD(n) =$ the number of non-isomorphic rank 2 matroids on n elements (can be found in [1])

The following observations from the given table might be worth noticing:

The number of "greedoidic" graphs is considerably small when compared with the number of all the simple graphs (it is very likely that almost all simple graphs are not "greedoidic"). It seems that the figures for $GG(n+1)$, $GD(n+1)$ and $GC(n+1)$ are approximately three times larger than the figures for $GG(n)$, $GD(n)$, $GC(n)$ respectively, as well as that the ratios $GD(n)/GG(n)$ and $GC(n)/GD(n)$ are considerably stable.

6. Graph-theoretic approach

The non-trivial greedoid axioms (2) and (3) can be translated to graphs with marked vertices as follows:

(2'): Each edge has at least one marked vertex.

(3'): Each marked vertex is connected by an edge to at least one vertex of each non-incident edge.

Statement 1. *Simple graphs adjoined to full rank 2 greedoids must be connected and their diameter is less than 4.*

Proof. Non-marked isolated vertices would correspond to non-full cases. Marked isolated vertices would contradict (3'). Thus each connected component contains an edge. By (2') each such edge contains a marked vertex. The existence of an edge in another connected component contradicts (3'). Quite a similar proof applies to the case when we have two vertices, the shortest path between which includes at least four edges. The vertices of the first and the last edge of such a path are not connected by an edge and they can be treated as if they belonged to different connected components. □

This statement can be used for a considerable reduction of input data; we can restrict ourselves to the graphs with the described property.

We point out that the brute-force search for non-isomorphic rank 2 greedoids from the "bottom" (this means: starting from the family of feasible 1-sets, we test all the possibilities for the families of feasible 2-sets) would require the testing of

$$n \cdot 2^{\binom{n}{2}}$$

such families. Thus, given $n = 6$ and $n = 7$ we would have to test 196608, respectively 7340032 families in order to finally keep only 200, respectively 592 of them, in our list of non-isomorphic greedoids. The corresponding number of tests (effectively performed by our computer) with our "roof" approach is equal to only $SG(n) \cdot 2^n$, that is, 9984 and 133632 for $n = 6$ and $n = 7$ respectively.

The factor $SG(n)$ can be replaced by the number of non-isomorphic graphs satisfying Statement 1. This would lead to a futher reduction of the number of necessary tests. Note, however, that it is hard to use Statement

1 directly if we want to use complementation at the same time: the complements of disconnected graphs are connected and the complements of graphs with diameter greater than 3 have a diameter even less than 3.

If we decide to abandon using complementations, then we can restrict ourselves to listing the full greedoids when on the ground-set of a given cardinality; the non-full greedoids can be obtained by bijection from the full greedoids on smaller ground-sets.

It is easy to give complete data for rank 2 greedoids arising from some special general classes of simple graphs:

Statement 2. *The total number of non-isomorphic rank 2 greedoids on n elements with associated trees is equal to*

$$\frac{1}{4} \cdot (n^2 - 2n - 9 + \frac{1 + (-1)^n}{2})$$

Proof. According to Statement 1., it is easy to realize that the only possible "greedoidic" trees are stars and double stars (two stars which have one edge in common; each of them has at least one additional edge).

Given a star with edges $12, 13, \ldots, 1k$, $(k > 2)$, it is obvious that there are $k + 1$ non-isomorphic possibilities for the (regularly) marked subsets of vertices:

$$1, 12, 123, \ldots, 123 \ldots k, 23 \ldots k$$

Specially, there are only two non-isomorphic possibilities for $k = 2$. Thus the total number of possibilities (including non-full cases) is equal to

$$2 + (4 + 5 + \ldots + (n + 1)) = \frac{1}{2} \cdot (n^2 + 3n - 6).$$

Given a double star with the common edge 12 connecting the two non-trivial substars, it is obvious that 12 is the only possible regularly marked subset of vertices. If the double star is incident to k vertices, there are $\lfloor (k-2)/2 \rfloor$ non-isomorphic possibilities with respect to the number of legs in the two substars (note that we should not make a difference between the substars). It follows that the total number of possibilities is equal to

$$\lfloor 2/2 \rfloor + \lfloor 3/2 \rfloor + \ldots + \lfloor (n-2)/2 \rfloor = \frac{1}{4} \cdot (n^2 - 4n + 3 + \frac{1 + (-1)^n}{2})$$

The statement is derived by summing up the last two expressions. □

Remark. The star on n vertices has the maximal number (equal to $2^n + 1$) of associated greedoids (including isomorphic copies), within the class of all the non-isomorphic simple graphs on n vertices.

Let K_n and A_n respectively denote the complete graph on n vertices and K_n with one edge deleted.

Statement 3. *For each $n \geq 2$ there are two non-isomorphic greedoids associated to K_n. For each $n \geq 3$ there are four non-isomorphic greedoids associated to A_n.*
Proof. If a vertex of K_n is not marked, then by (2') all its $n-1$ neighbours must be marked. Another possibility is that all n vertices are marked. Let A_n be obtained from K_n by deletion of the edge xy and let z be an arbitrary third vertex. The four non-isomorphic associated greedoids have the non-marked sets of vertices $\emptyset$, x, xy, z respectively. □

Final remarks: Our construction of rank 2 greedoids is a combination of the computer approach and "hand" approach. We did not use the computer for difficult isomorphism tests (either on graph or on greedoid level). Our intuition seems to be more efficient for this purpose, at least for small-size examples, especially when the search for orbits is considered. However, we did use the computer for quick and routine (one-loop and double-loop respectively) checkings of (2) and (3); it would be almost hopeless to perform these checkings by hand. Such a combined approach seems to be optimal in this case; the computer work and the "hand" work successfully complement each other. It might be interesting to look for other examples of such a complementation.

References

[1] Acketa, D. M., On the enumeration of matroids of rank 2, Univ. u Novom Sadu Zb. Rad. Prirod. - Mat. Fak. Ser. Mat., 8 (1978), 83-90.

[2] Acketa, D. M., A construction of all non-isomorphic greedoids on at most 5 elements, Univ. u Novom Sadu Zb. Rad. Prirod. - Mat. Fak. Ser. Mat., 18 (1988), 93-99.

[3] Harary, F., Graph Theory, Addison-Wesley, 1973.

[4] Harary, F., Palmer, E.M., Graphical Enumeration, Academic Press, 1973.

[5] Korte, B., Lovasz, L., Greedoids, a structural framework for the greedy algorithm, in W. R. Pulleyblank (ed.): Progress in Combinatorial Optimization, Proceedings of the Silver Jubilee Conference on Combinatorics, Waterloo, Academic Press, 1983.

REZIME

O GRIDOIDIMA RANGA 2

Gridoidima ranga 2 su pridruženi specijalni grafovi sa označenim čvorovima. Ova reprezentacija je korišćena za generisanje liste svih neizomorfnih gridoida na nosačima od najviše 7 elemenata. Pri generisanju je delimično korišćen računar.

Received by the editors January 19, 1989