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PARTIALLY ORDERED AND RELATIONAL VALUED ALGEBRAS AND CONGRUENCES

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Abstract

Different kinds of fuzzy subalgebras of an algebra (lattice valued, partially ordered, relational) are defined as the mappings from an algebra to the corresponding structure (lattice, partially ordered set, relational structure), with the property that every level subset is an ordinary subalgebra. Fuzzy congruences are introduced similarly. Every fuzzy subalgebra of an algebra is uniquely determined by a suitable collection of subalgebras and vice versa (the same is with congruences). It is proved that all lattice valued subalgebras (congruences) of an algebra, form an algebraic lattice, independant of the lattice being the codomain of the mappings. The collection of partially ordered and relational valued subalgebras is also uniquely determined by the algebra itself. Some consequences in the case of fuzzy subgroups and normal fuzzy subgroups of a group are investigated. In particular, conditions under which the lattice of all lattice valued fuzzy subgroups is Boolean are given.

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1. Preliminaries

If $(P, \leq)$ is a partially ordered and A a nonempty set, then any mapping $\overline{A} : A \to P$ is a **partially ordered fuzzy set (P-fuzzy set)** on A (see [1]). A **p-cut** of A, for $p \in P$, is a mapping $\overline{A} : A \to \{0,1\}$, such that for $x \in A$, $\overline{A}(x) = 1$ iff $A(x) \geq p$. Obviously, $\bar{A}_p$ is the characteristic function of the following subset of A, called also a p-cut or a level subset of $\overline{A}: A_p = \{x \in A | \bar{A}(x) \geq p\}$.

If a partially ordered set is a complete lattice L with the bottom element 0 and the top element 1, then the corresponding fuzzy set $\overline{A} : A \to L$ is said to be **lattice valued (L-valued).**

Some elementary properties of L- and P-fuzzy sets are the following (Propositions I to VI) (for the proofs, see the papers listed in References).

I If $\overline{A}$ is an L-fuzzy set on A, then for every $x \in A$

$$\overline{A}(x) = \bigvee_{p \in L} p \circ \overline{A}_p(x),$$

where

$$p \circ \bar{A}_p(x) = \begin{cases} p, & \text{if } \bar{A}_p(x) = 1 \\ 0, & \text{otherwise.} \end{cases}$$

II Let $\bar{A}$ be an L-fuzzy set on A. Then

(a) The collection $\{A_p | p \in L\}$ of level subsets of $\bar{A}$ is closed under intersections and contains A, hence it is a Moore's family of subsets of A, and a lattice under $\subseteq$;

(b)

$$\bigcap_{p \in K \subseteq L} A_p = A_{\bigvee_{p \in K} p};$$

(c) $p \leq q$ implies $A_q \subseteq A_p$.

III Let A be a finite set and F a family of its subsets closed under arbitrary intersections and containing A (the Moore's family of subsets of A). Let also L be the lattice dual to $(F, \subseteq)$ and $\bar{A} : A \to L$ an L-valued subset of A defined with

$$\bar{A}(x) := \bigcap \{f \in F | x \in f\}.$$

Then, the lattice of level subsets of $\bar{A}$ is isomorphic with $(F, \subseteq)$.

IV If $\bar{A} : A \to P$ is a P-fuzzy set on A, then for $x \in A$

$$\bar{A}(x) = \bigvee(p \in P | \bar{A}_p(x) = 1)$$

(i.e. the supremum on the right exists in $(P, \leq)$ for every $x \in A$ and is equal to $\bar{A}(x)$).

V Let $\bar{A} : A \to P$ be a P-fuzzy set on A. Then,

a) if $p, q \in P$ and $p \leq q$, then for every $x \in A$, $\bar{A}_q(x) \leq \bar{A}_p(x)$;

b) if for $Q \subseteq P$ there exists a supremum of Q $(\bigvee(p|p \in Q))$, then

$$\bigcap(A_p | p \in Q) = A_{\bigvee(p|p\in Q)};$$

c) $\bigcup(A_p | p \in P) = A$;

d) for every $x \in A$, $\bigcap(A_p | x \in A_p)$ is a level-subset of $\bar{A}$.

VI Let P be a family of subsets of a nonempty set A, union of which is also A, and such that for every $x \in A$, $\bigcap(p \in P | x \in p) \in P$. Let $\bar{A} : A \to P$ be defined with

$$\bar{A}(x) := \bigcap(p \in P | x \in p).$$

Then, $\bar{A}$ is a P-fuzzy set on A, where $(P, \leq)$ is a dual of $(P, \subseteq)$, and for every $p \in P$, $p = A_p$.

Let $\mathcal{A} = (A, F)$ be an algebra and $(P, \leq)$ a partially ordered set. Any mapping $\bar{A} : A \to P$ such that for every $p \in P$ (A_p, F) is a subalgebra of $\mathcal{A}$ is said to be a P-**valued subalgebra** of $\mathcal{A}$. A_p is the **level-subalgebra** of $\bar{A}$. A mapping $\overline{\rho} : A^2 \to P$ is a P-**valued congruence relation** on A if for every $p \in P$, ρ_p (the **level-congruence**) is an ordinary congruence on A.

Lattice valued subalgebras and congruences can be explicitely described in the following way (Propositions VII and VIII).

VII If $\mathcal{A} = (A, F)$ is an algebra and L a complete lattice, then an L-fuzzy subalgebra of $\mathcal{A}$ is a fuzzy subset $\bar{A} : A \to L$ of A, such that for all $x_1, ..., x_n \in A$ and every n-ary operation f from F

$$A(f(x_1, ..., x_n)) \geq \bar{A}(x_1) \wedge ... \wedge \bar{A}(x_n)$$

and for every constant c from $\mathcal{A}$, $\bar{A}(c) = 1$.

VIII A mapping $\overline{\rho} : A^2 \to L$ is an L-valued congruence relation on $\mathcal{A} = (A, F)$ if it is an L-valued relation on A, satisfying the following:

(i) $\overline{\rho}$ is reflexive, i.e. $\overline{\rho}(x,x) = 1$, for all $x \in A$;

(ii) $\overline{\rho}$ is symmetric, i.e. for all $x, y \in A$ $\overline{\rho}(x,y) = \overline{\rho}(y,x)$;

(iii) $\overline{\rho}$ is transitive, i.e. for all $x, y, z \in A$

$$\overline{\rho}(x,y) \geq \overline{\rho}(x,z) \wedge \overline{\rho}(z,y);$$

(iv) $\overline{\rho}$ is compatible with the operations in $\mathcal{A}$, i.e. for every $f \in F_n \subseteq F$, for all $x_1, ..., x_n, y_1, ..., y_n \in A$,

$$\overline{\rho}(f(x_1, ..., x_n), f(y_1, ..., y_n)) \geq \bigwedge_{i=1}^{n} \overline{\rho}(x_i, y_i).$$

VII and VIII could be taken as the definitions of L-valued subalgebras and congruences, in which case one could easely prove that every level subset is an ordinary subalgebra (or congruence) of $\mathcal{A}$.

Some other basic properties of L-valued algebraic structures can be found in References.

2. Lattice valued and partially ordered algebras and congruences

The aim of this part is to investigate the structure of the collection of all L- or P-valued subalgebras (congruences) of an algebra. First of all, we advance four propositions, claiming that every fuzzy set in that collection is uniquely determined by the particular family of ordinary subalgebras or congruences.

Proposition 1. *If $\bar{A} : A \to L$ is an L-valued subalgebra of $\mathcal{A} = (A, F)$, then the collection $\{A_p | p \in L\}$ of its level subalgebras is a Moore's family of subsets of A, and thus it is a lattice under the set inclusion.*

Proof. Straightforward, by II. □

Proposition 2. *Let $\mathcal{F} = \{B_i, i \in I\}$ be a Moore's family of subalgebras of $\mathcal{A} = (A, F)$, and let $L = (\mathcal{F}, \leq)$ be the lattice dual to $(\mathcal{F}, \subseteq)$. Then $\bar{A} : A \to \mathcal{F}$, such that for $x \in A$*

$$\bar{A}(x) = \bigcap(\mathcal{B} \in \mathcal{F} | x \in B)$$

is an L-valued subalgebra of $\mathcal{A}$. Moreover, for $\mathcal{B} \in \mathcal{F}$, $\bar{A}_{\mathcal{B}} = \mathcal{B}$.

Proof. Follows from III and from the definition of an L-valued subalgebra. □

Properties of subalgebras determinig a P-valued one are the following.

Proposition 3. *Let $\bar{A} : A \to P$ be a P-valued subalgebra of $\mathcal{A} = (A, \mathcal{F})$. Then the union of all the level-subalgebras of $\bar{A}$ is A, and for every $x \in A$, the intersection of all the level subalgebras containing it is also a level-subalgebra of $\bar{A}$.*

Proof. By V, c) and d). □

Proposition 4. *Let $\mathcal{A} = (A, \mathcal{F})$ be an algebra, and $\mathcal{F}$ the family of its subalgebras (including possibly the empty set) union of which is A, and such that for every element of A the intersection of subalgebras in $\mathcal{F}$ containing it belongs to $\mathcal{F}$. Let $P = (\mathcal{F}, \leq)$ be the dual of $(\mathcal{F}, \subseteq)$. Then, the function $\bar{A} : A \to P$, defined with $\bar{A}(x) := \bigcap(\mathcal{B} \in \mathcal{F} | x \in \mathcal{B})$ is a P-valued subalgebra of $\mathcal{A}$, and the collection of level-subalgebras of $\mathcal{A}$ coincides with $\mathcal{F}$.*

Proof. Straightforward, by VI. □

The above construction (precisely the one from Proposition 2) can be applied on an arbitrary collection af algebras.

Theorem 1. *Let $\{\mathcal{A}_i, i \in I\}$ be a collection of algebras, each having a trivial (one-element) subalgebra. Then, there is an algebra $\mathcal{A}$ and its lattice valued subalgebra $\bar{A}$, with the lattice of level-subalgebras represented in Fig.1, and such that F_i are isomorphic with $\mathcal{A}_i$.*

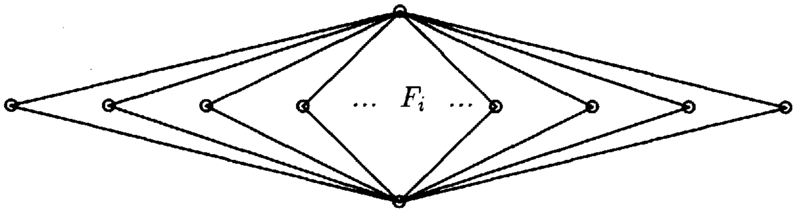

Fig.1

Proof. Set $\mathcal{A} = \prod(\mathcal{A}_i|i \in I)$. Each $\mathcal{A}_i$ is isomorphic with a subalgebra of $\mathcal{A}$, namely with F_i , all projections of which are trivial, except the i-th, which is equal to $\mathcal{A}_i$. Let $\mathcal{E}$ be the trivial, one-element algebra, taken to be a subalgebra of $\mathcal{A}_i$ for all $i \in I$. The collection $\mathcal{F} = \{\mathcal{E}, \mathcal{A}\} \bigcup \{F_i|i \in I\}$ is a Moore's family of subalgebras of $\mathcal{A}$, represented as a lattice in Fig.1. Applying Proposition 2, we finally obtain a lattice valued subalgebra $\bar{A}$: $A \to \mathcal{F}$ of $\mathcal{A}$, such that its level subalgebras coincide with the algebras from $\mathcal{F}$. Clearly, $\mathcal{F}_i$ is for every $i \in I$ isomorphic with $\mathcal{A}_i$. □

Since congruences are subalgebras of $\mathcal{A}^2$, it is clear that similar propositions concerning L- and P-valued congruences are also valid, and we shall not formulate them separately.

The following relation on the collection of P-valued subalgebras as functions from an algebra to arbitrary posets, equalizes fuzzy subalgebras which have the same set of level subalgebras.

Let $\bar{A}_1 : A \to P_1$ and $\bar{A}_2 : A \to P_2$ be two partially ordered subalgebras of the same algebra $\mathcal{A} = (A, \mathcal{F})$, where $(P_1, \leq)$ and $(P_2, \leq)$ are two arbitrary posets. We shall say that $\bar{A}_1$ and $\bar{A}_2$ are P-equivalent, and denote it with $\bar{A}_1 \Leftrightarrow_P \bar{A}_2$, if

$$\{(A_1)_p|p \in P_1\} = \{(A_2)_p|p \in P_2\}.$$

If P_1 and P_2 are lattices, then $\bar{A}_1$ and $\bar{A}_2$ are said to be L-equivalent $(\bar{A}_1 \Leftrightarrow_L \bar{A}_2)$. Evidently, $\Leftrightarrow_P$ is an equivalence relation on the collection

$$\bigcup(Sub_P\mathcal{A}|P \in \mathcal{P}),$$

where P runs over a class $\mathcal{P}$ of all posets, and for a particular poset P, $Sub_P\mathcal{A}$ is a set of all P-valued subalgebras of $\mathcal{A}$:

$$Sub_P\mathcal{A} := \{\bar{A}|\bar{A} : A \to P, \text{ for every } p \in P, A_p \text{ is a subalgebra of } \mathcal{A}\}$$

(for a lattice L, we have the set $Sub_L\mathcal{A}$). Functions from $\bigcup Sub_P\mathcal{A}$ can be ordered in a natural way: if $\bar{A}_1 : A \to P_1$, $\bar{A}_2 : A \to P_2$, then

$$(1) \qquad \bar{A}_1 \ll \bar{A}_2 \text{ iff } \{(A_1)_p | p \in P_1\} \subseteq \{(A_2)_p | p \in P_2\}.$$

The above relation ($\ll$) is obviously a quasiorder (reflexive and transitive relation) on $\bigcup Sub_P\mathcal{A}$.

Set $Sub_{\mathcal{P}}\mathcal{A} :\!= \bigcup(Sub_P\mathcal{A} | P \in \mathcal{P})/_{\Leftrightarrow_P}$ ($Sub_{\mathcal{L}}\mathcal{A}$ in the case of lattices).

Since $\bar{A}_1 \Leftrightarrow_P \bar{A}_2$ iff $\bar{A}_1 \leq \bar{A}_2$ and $\bar{A}_2 \leq \bar{A}_1$, the relation $\ll$ induces an ordering on $Sub_{\mathcal{P}}\mathcal{A}$ (on $Sub_{\mathcal{L}}\mathcal{A}$):

$$(2) \qquad [\bar{A}_1]_{\Leftrightarrow} \leq [\bar{A}_2]_{\Leftrightarrow} \text{ iff } \bar{A}_1 \ll \bar{A}_2.$$

By the above construction, we can identify each fuzzy subalgebra of $\mathcal{A}$ with the appropriate collection of ordinary subalgebras of the same algebra, and by (1), the ordering $\leq$ in (2) is in fact the set inclusion among these collections.

Theorem 2. *For an algebra $\mathcal{A}$, $Sub_{\mathcal{L}}\mathcal{A}$ is a complete lattice under the relation $\leq$ from (2).*

Proof. We shall show that $Sub_{\mathcal{L}}\mathcal{A}$ is closed under arbitrary intersections and that it contains the greatest element.

Indeed, $Sub_{\mathcal{L}}\mathcal{A}$ consists of Moore's families of ordinary subalgebras of $\mathcal{A}$ (each of them closed under intersections and containing $\mathcal{A}$). An arbitrary intersection of Moore's families is again a Moore's family of subalgebras, and since all the subalgebras of $\mathcal{A}$ also compose a Moore's family being a greatest element in $Sub_{\mathcal{L}}\mathcal{A}$, we are done. □

Example 1.

The lattice $Sub_{\mathcal{L}}\mathcal{G}$ of the Klein's group G is given by its Hasse diagram in Fig.2.

o	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

The Moore's families representing L-valued subgroups of $\mathcal{G}$ are: $O = \{\mathcal{G}\}$, $A = \{\{e\},\mathcal{G}\}$, $B = \{\{e,a\},\mathcal{G}\}$, $C = \{\{e,b\},\mathcal{G}\}$, $D = \{\{e,c\},\mathcal{G}\}$, $E = \{\{e\},\{e,a\},\mathcal{G}\}$, $F = \{\{e\},\{e,b\},\mathcal{G}\}$, $G = \{\{e\},\{e,c\},\mathcal{G}\}$, $H = \{\{e\},\{e,a\},\{e,b\},\mathcal{G}\}$, $I = \{\{e\},\{e,a\},\{e,c\},\mathcal{G}\}$, $J = \{\{e\},\{e,b\},\{e,c\},\mathcal{G}\}$, $1 = \{\{e\},\{e,a\},\{e,b\},\{e,c\},\mathcal{G}\}$.

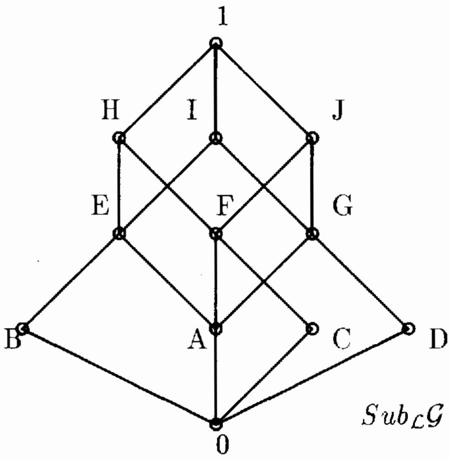

Fig.2

Each of the above listed families determines a lattice L and an L-valued subgroup of $\mathcal{G}$. Moreover, these are the only lattice valued subgroups of the Klein's group: for any lattice L and a function $\bar{A}$ from $\mathcal{G}$ to L which is a lattice-valued subgroup of $\mathcal{G}$, the set of its level-subgroups coincides with a family from $Sub_{\mathcal{L}}\mathcal{G}$. For example, the family $F = \{\{e\},\{e,b\},\mathcal{G}\}$ determines (by virtue of Proposition 2) an L-valued subgroup $\bar{A} : \mathcal{G} \to L$, where

$$\bar{A} = \begin{pmatrix} e & a & b & c \\ \{e\} & \mathcal{G} & \{e,b\} & \mathcal{G} \end{pmatrix},$$

and L is the lattice given in Fig.3.

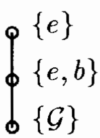

Fig.3

The same family F of level subgroups can be obtained if the lattice L is the real interval $[0,1]$, and

$$\bar{A} = \begin{pmatrix} e & a & b & c \\ 1 & 0 & \mathrm{p} & 0 \end{pmatrix},$$

where p is an arbitrary number between 0 and 1. However, not all the families from $Sub_{\mathcal{L}}\mathcal{G}$ could be obtained by real valued functions (the family H, for instance).

Lemma 1. *For an algebra $\mathcal{A}$, the lattice $Sub_{\mathcal{L}}\mathcal{A}$ is atomically generated.*

Proof. Every single subalgebra is by itself an one-element Moore's family of subalgebras. □

Theorem 3. *The lattice $Sub_{\mathcal{L}}\mathcal{A}$ of lattice valued subalgebras of an algebra $\mathcal{A}$ is algebraic.*

Proof. Follows by Lemma 1, since every $F \in Sub_{\mathcal{L}}\mathcal{A}$ is a family of subalgebras - atoms in that lattice. □

As a consequence of P-equivalency, every partially ordered subalgebra of an algebra $\mathcal{A}$ can be identified with the collection of ordinary subalgebras satisfying conditions from Proposition 4. However, the set of all such collections, $Sub_{\mathcal{P}}\mathcal{A}$, is not generally a lattice under the ordering $\leq$ introduced by (2), i.e. under the set inclusion among collections of subalgebras.

Example 2. The poset $Sub_{\mathcal{P}}\mathcal{G}$ of partially ordered subgroups of the Klein's group is represented in Fig.4. The family in which $Sub_{\mathcal{P}}\mathcal{G}$ differs from the lattice $Sub_{\mathcal{L}}\mathcal{G}$ is $K = \{\{e\},\{e,a\},\{e,b\},\{e,c\}\}$.

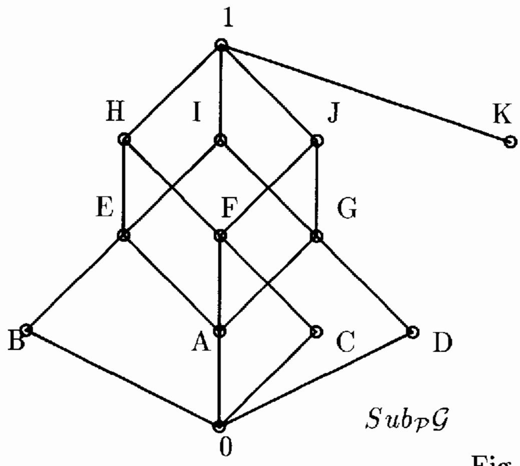

Fig.4

If congruences of an algebra are considered as reflexive, symmetric and transitive subalgebras of $\mathcal{A}^2$, then obviously definitions (1) and (2) concerning P- and L- equivalency can be directly applied on fuzzy congruences. Hence, for a particular poset P or a lattice L, $Con_P\mathcal{A}$ is a set of all P-valued and $Con_L\mathcal{A}$ a set of all L-valued congruences of $\mathcal{A}$. Consequently,

$$Con_{\mathcal{P}}\mathcal{A} := \bigcup(Con_P\mathcal{A}|P \in \mathcal{P})/\Leftrightarrow_P$$

$$Con_{\mathcal{L}}\mathcal{A} := \bigcup(Con_L\mathcal{A}|L \in \mathcal{L})/\Leftrightarrow_L,$$

where $\mathcal{P}$ is a class of all posets, $\mathcal{L}$ a class of lattices, and $\Leftrightarrow_P$ and $\Leftrightarrow_L$ relations of P- and L- equivalency for fuzzy subalgebras of $\mathcal{A}$.

From the preceding consideration and by Theorems 2, 3 and Lemma 1, it follows immediately that the following theorem holds.

Theorem 4. *For an algebra $\mathcal{A}$, $Con_{\mathcal{L}}\mathcal{A}$ is an atomically generated algebraic lattice under the relation $\leq$ from (2).* □

3. Relational valued subalgebras and congruences

An R-**fuzzy set** on X is a mapping $\bar{A} : X \to S$ where X is a nonvoid set, $\mathcal{S}(S, R)$ a relational system, and R is a binary relation on $S \neq \emptyset$.

For every $p \in S$, the **p-cut** of $\bar{A}$ is a mapping $\bar{A}_p : A \to \{0,1\}$, such that for $x \in A$, $\bar{A}_p(x) = 1$ iff $(p, \bar{A}(x)) \in R$. $\bar{A}_p$ is the characteristic function of the following subset of A:

$$A_p := \{x \in A | \bar{A}_p(x) = 1\}.$$

Let $\mathcal{S} = (S, R)$ be a relational system such that for all $a, b \in S$

$$(3) \qquad a \neq b \text{ implies } \{x \in S | (x, a) \in R\} \neq \{x \in S | (x, b) \in R\}.$$

R is said to have the **unique projection property**, briefly R is a **UP-relation**, if it satisfies (3). In that case we say that $\mathcal{S}$ is a **UP-relational system**.

An R-fuzzy set $\bar{A} : A \to S$ enables a unique decomposition and synthesis into the family of p-cuts if and only if (S, R) is a UP-relational system. This was given in [2], where the following proposition was also (implicitely) proved.

Proposition 5. *For any collection $\mathcal{F}$ of subsets of a nonempty set X there is a UP-relational system (S, R) and an R-fuzzy set $\bar{A} : X \to S$ such that $\mathcal{F}$ is its family of level-subsets.* □

Let $\mathcal{A} = (A, \mathcal{F})$ be an algebra and (S, R) a UP-relational system. The mapping $\bar{A} : A \to S$ is a **relational valued subalgebra** (R- fuzzy subalgebra) of $\mathcal{A}$ if for every $p \in S$, (A_p, F) is a subalgebra of $\mathcal{A}$. The mapping $\overline{\rho} : A^2 \to S$ is a **relational valued congruence** on $\mathcal{A}$ if every $\overline{\rho}_p$ is a congruence on $\mathcal{A}$.

As a consequence of Proposition 5, we have the following theorem.

Theorem 5. [4] *For any collection $\mathcal{F}$ of subalgebras of an algebra $\mathcal{A}$, there is an R-valued fuzzy subalgebra of $\mathcal{A}$, such that $\mathcal{F}$ is its family of p-cuts.* □

Just as in the case of P- and L-valued subalgebras, families of ordinary subalgebras determine relational valued ones. To show this, we need the notion of R-equivalency.

Let $\bar{A}_1 : A \to S_1$ and $\bar{A}_2 : A \to S_2$ be two relational valued subalgebras of the same algebra $\mathcal{A} = (A, \mathcal{F})$, where (S_1, R_1) and (S_2, R_2) are two UP-relational systems. We shall say that $\bar{A}_1$ and $\bar{A}_2$ are **R-equivalent**, and denote it with $\bar{A}_1 \Leftrightarrow_R \bar{A}_2$, if

$$\{(A_1)_p | p \in S\} = \{(A_2)_p | p \in S\}.$$

Again, $\Leftrightarrow$ is an equivalence relation, this time on the collection $\bigcup(Sub_S\mathcal{A} | S \in \mathcal{R})$, where S runs over a class $\mathcal{R}$ of all UP-relational systems, and for a particular system (S, R), $Sub_S\mathcal{A}$ is a set of all R-valued subalgebras of $\mathcal{A}$:

$$Sub_S\mathcal{A} := \{\bar{A} | \bar{A} : A \to S, \text{ for every } p \in S,\ A_p \text{ is a subalgebra of } \mathcal{A}\}.$$

By virtue of Theorem 5, $Sub_S\mathcal{A}$ consists of all functions from A to S. All these functions from $\bigcup Sub_S\mathcal{A}$ can also be (quasi)ordered: if $\bar{A}_1 : A \to S_1$, $A_2 : A \to S_2$, then

$$\bar{A}_1 \ll \bar{A}_2 \text{ iff } \{(A_1)_p | p \in S_1\} \subseteq \{(A_2)_p | p \in S_2\}.$$

Set $Sub_{\mathcal{R}}\mathcal{A} := \bigcup(Sub_S\mathcal{A} | S \in \mathcal{R}) / \Leftrightarrow_R$.

The following is an ordering on $Sub_{\mathcal{R}}\mathcal{A}$:

$$[\bar{A}_1]_{\Leftrightarrow} \leq [\bar{A}_2]_{\Leftrightarrow} \text{ iff } \bar{A}_1 \ll \bar{A}_2.$$

Theorem 6. *For an algebra $\mathcal{A}$, the set $Sub_{\mathcal{R}}\mathcal{A}$ is under $\leq$ a Boolean algebra isomorphic with $P(Sub\mathcal{A})$ (i.e. with the power set of the collection of all ordinary subalgebras of $\mathcal{A}$).*

Proof. Every relational valued subalgebra of $\mathcal{A}$ can be identified with a collection of ordinary subalgebras of $\mathcal{A}$, and vice versa (by Theorem 5), and the relation $\leq$ coincides with set inclusion among these collections. Thus every subset of $Sub\mathcal{A}$ determines exactly one relational valued subalgebra of $\mathcal{A}$. □

Example 3. [4]

Let $\mathcal{G}$ be a Klein's group, and let (S, R) be given by its graph (Fig.5).

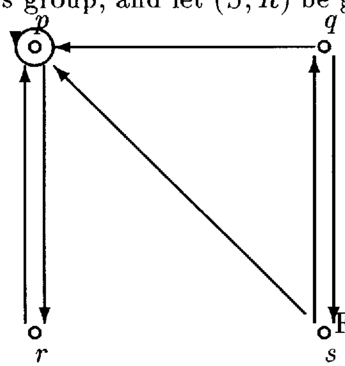

Fig.5

Let also $\bar{A} : G \to S$ be given with

$$\bar{A} = \begin{pmatrix} e & a & b & c \\ \mathrm{p} & \mathrm{q} & \mathrm{r} & \mathrm{s} \end{pmatrix}.$$

$\bar{A}$ is an R-vàlued subgroup of $\mathcal{G}$. Indeed, all the p-cuts are subgroups of $\mathcal{G}$:

$$A_p = \{e, b\};\ A_q = \{e, c\};\ A_r = \{e\};\ A_s = \{e, a\}.$$

Moreover, every subset of the subgroup lattice of $\mathcal{G}$ determines an R-valued subgroup of $\mathcal{G}$. For example, the subgroups $H = \{e, a\}$ and $K = \{e, b\}$, by the construction described in [4], determine a relational system (S, R) (Fig.6), and a mapping $\bar{A}$:G$\to S$, which is an R-valued subgroup of $\mathcal{G}$. Indeed, the characteristic functions of the above subgroups are

$$\bar{A} = \begin{pmatrix} e & a & b & c \\ 1 & 1 & 0 & 0 \end{pmatrix} \text{ and } \bar{A} = \begin{pmatrix} e & a & b & c \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

Hence, we have the following function and a collection of p-cuts:

$$S = \{p, q, r, s\},\ \bar{A} = \begin{pmatrix} e & a & b & c \\ \mathrm{p} & \mathrm{q} & \mathrm{r} & \mathrm{s} \end{pmatrix}.$$

$\bar{A}$	e	a	b	c
	p	q	r	s
$\bar{A}_p$	1	1	0	0
$\bar{A}_q$	1	0	1	0
$\bar{A}_r$	1	0	1	0
$\bar{A}_s$	1	0	1	0

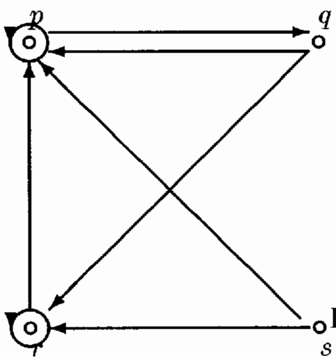

Fig.6

4. Application to groups

All the results about arbitrary algebras and their fuzzy subalgebras could be easely applied on groups, taken to have one binary, one unary and a nullary operation, as it has been done in the above examples. However, some particular properties of fuzzy subgroups deserve to be mentioned.

Recall that a function $\bar{A}: G \to P$ of a group $\mathcal{G}$ into the poset P is its P-valued subgroup, if all level subsets of $\bar{A}$ are subgroups of $\mathcal{G}$. If P is a lattice, then $\bar{A}$ is an L-valued subgroup of $\mathcal{G}$. Similar definition can be given for a relational valued subgroup of $\mathcal{G}$. An alternative definition of an L-valued subgroup is the following: it is a mapping $\bar{A}$ of a group $(G, \circ, ^{-1}, e)$ into the lattice L, such that $\bar{A}(e) = 1$, $\bar{A}(x \circ y) \geq \bar{A}(x) \wedge \bar{A}(y)$, and $\bar{A}(x^{-1}) \geq \bar{A}(x)$, for all $x, y \in G$. In all the above cases, $\bar{A}$ is said to be **normal**, if all its level subgroups are normal.

Proposition 6. *The collection $Sub(n)_{\mathcal{L}}\mathcal{G}$ of all L-valued normal subgroups of a group $\mathcal{G}$ is a lattice, a sublattice of $Sub_{\mathcal{L}}\mathcal{G}$.*

Proof. The lattice of normal subgroups of $\mathcal{G}$ is a sublattice of $Sub\mathcal{G}$, and every Moore's family is closed under intersections. Hence, the set of all Moore's families of normal subgroups is a sublattice of $Sub_{\mathcal{L}}\mathcal{G}$. □

Theorem 7. *For a finite group $\mathcal{G}$, $Sub_{\mathcal{L}}\mathcal{G}$ is a Boolean lattice with $2^{|Sub\mathcal{G}|-1}$ elements if and only if $\mathcal{G}$ is cyclic and $|G| = p^k$ (p-prim).*

Proof. If $\mathcal{G}$ is cyclic and $|G| = p^k$, then $Sub G$ is a chain (with $k+1$ element), every nonempty subset of which is its sublattice and a Moore's family, provided that it contains G. Hence, $Sub_{\mathcal{L}}\mathcal{G}$ is a Boolean lattice with 2^k elements. On the other hand, if $Sub_{\mathcal{L}}\mathcal{G}$ is Boolean and $|Sub_{\mathcal{L}}\mathcal{G}| = 2^{|Sub\mathcal{G}|-1}$, then every subset of $Sub\mathcal{G}$ containing G is a Moore's family and its sublattice, proving that $Sub\mathcal{G}$ is a chain. Hence, $\mathcal{G}$ is a cyclic group with $2^{|Sub\mathcal{G}|-1}$ elements.□

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REZIME

PARCIJALNO-UREDJENE I RELACIONO VREDNOSNE ALGEBRE I KONGRUENCIJE

U radu se definišu parcijalno-uredjene i relaciono vrednosne algebre i kongruencije preko nivo skupova, koji moraju biti obične podalgebre, odnosno kongruencije neke algebre. Pokazano je da su mrežno-vrenosne podalgebre jednoznačno odredjene Murovskim familijama podalgebri date algebre, a parcijalno uredjene podalgebre familijama podalgebri zatvorenim za preseke po koordinatama karakterističnih funkcija. Relaciono-vrednosne podalgebre odredjene su proizvoljnim, nepraznim familijama običnih podalgebri. Kolekcija mrežno-vrednosnih podalgebri je algebarska mreža u odnosu na inkluziju. Slična tvrdjenja vaze i za kongruencije. Posebno su razmatrane grupe. Dati su potrebni i dovoljni uslovi da mreža mrežno-vrednosnih podgrupa konačne grupe bude Bulova.

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