Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 23, 2 (1993), 233 - 237

Review of Research
Faculty of Science
Mathematics Series

ANTI-INVERSE BOOLEAN SEMIGROUPS

Koriolan Gilezan
Institute of Mathematics, University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

In this paper the necessary and sufficient condition for Boolean matrix to belong to an anti-iverse Boolean semigroup is given.

AMS Mathematics Subject Classification (1991): 47A60
Key words and phrases: anti-inverse Boolean semigroups, Boolean algebra, matrices, boolean equations.

An $n \times n$ matrix $A = [a_{ij}]$ is a Boolean matrix if its elements are Boolean trems from the Boolean algebra $(B, \cup, \cdot, -)$, $B = \{0, 1\}$. M_n is the set of Boolean matrices of the same dimension $n \times n$. Let a binary operation "$\circ$" defined on M_n

$$A \circ B = [\bigcup_{k=1}^{n} a_{ik} \cdot b_{kj}], \quad i, j = 1, \dots, n$$

$(M_n, \circ)$ is a semigroup. Let $S_n \subset M_n$, $(S_n, \circ)$ be a subsemigroup of $(M_n, \circ)$.

A semigroup S_n is anti-inverse if for each $A \in S_n$ there exists such $X \in S_n$ that $A = X \circ A \circ X$ and $X = A \circ X \circ A$. i.e.:

(1) $(\forall A \in S_n)(\exists X \in S_n)(A = X \circ A \circ X \text{ and } X = A \circ X \circ A).$

This problem is considered in [1].

Theorem 1. *The statement (1) is equivalent to*

$$(2)\qquad (\forall A \in S_n)(\exists X \in S_n)(\overline{X} \cdot (A \circ X \circ A) \cup X \cdot (\overline{A \circ X \circ A}) \cup \overline{A} \cdot (X \circ A \circ X) \cup A \cdot (\overline{X \circ A \circ X}) = 0),$$

where

$$A \cdot B = [a_{ij} \cdot b_{ij}]$$
$$\overline{A} = [\overline{a_{ij}}], \quad i,j = 1, \dots, n.$$

Proof. It is known (see [2]) that Boolean equations $f = g$ and $\overline{f} \cdot g \cup f \cdot \overline{g} = 0$ are eqivalent. If we use this property in (1), it becomes (2)

$$(3)\quad (\forall A \in S_n)(\exists X \in S_n)(\overline{A} \cdot (X \circ A \circ X) \cup A \cdot (\overline{X \circ A \circ X}) = 0, \quad \overline{X} \cdot (A \circ X \circ A) \cup X \cdot (\overline{A \circ X \circ A}) = 0)$$

So (1) and (3) are equivalent statements. Further, (see [2]) the statement (3) becomes (2) applying the property that a system of Boolean equations $g_i = 0, \quad i = 1, \dots, m$ is equivalent to the single Boolean equation

$$\bigcup_{i=1}^{m} g_i = 0.$$

So Theorem 1 has been proved.□

Taking $A = [a_{ij}]$, $X = [x_{ij}]$ in (2) statement (1) becomes equivalent to the Boolean equation:

$$(4)\qquad \textstyle\bigcup_{i,j=1}^{n}[a_{ij}(\prod_{h=1}^{n}(\prod_{k=1}^{n}(\overline{a_{ik}} \cup \overline{x_{kh}}) \cup \overline{a_{hj}}) \cup \overline{a_{ij}}(\bigcup_{h=1}^{n}(\bigcup_{k=1}^{n} a_{ik}x_{kh})a_{hj} \cup x_{ij}(\prod_{h=1}^{n}(\prod_{k=1}^{n}(\overline{x_{ik}} \cup \overline{a_{kh}}) \cup \overline{x_{hk}}) \cup \overline{x_{ij}}(\bigcup_{h=1}^{n}(\bigcup_{k=1}^{n} x_{ik}a_{kh})x_{hj}] = 0$$

This has been the proof of the following theorem:

Theorem 2. *Statements (1) and (4) are equivalent.*

The left side of (4) is $f(x)$, where

$$x = (x_{11}, \dots, x_{1n}, x_{21}, \dots, x_{2n}, \dots, x_{n1}, \dots, x_{nn}).$$

It is known that each Boolean function f can be written in the canonical disjunctive form

$$(5) \qquad f(x) = \bigcup_{\alpha \in B^{n^2}} f_\alpha \cdot x_{11}^{\alpha_{11}} \cdots x_{1n}^{\alpha_{1n}} \cdots x_{n1}^{\alpha_{n1}} \cdots x_{nn}^{\alpha_{nn}},$$

where $\alpha = (\alpha_{11}, \ldots \alpha_{1n}, \alpha_{21}, \ldots \alpha_{2n}, \ldots \alpha_{n1}, \ldots \alpha_{nn})$.

Equation (5) has a solution if and only if

$$\prod_{\alpha \in B^{n^2}} f(\alpha) = 0$$

So the following theorem is proved.

Theorem 3. *A Boolean matrix A is the element of the antiinverse semigroup S_n if and only if the elements of A satisfy the condition:*

$$(6) \qquad \prod_{\alpha \in B^{n^2}} \{\bigcup_{i,j=1}^{n} [a_{ij}(\prod_{h=1}^{n}(\overline{a_{ik}} \cup \overline{\alpha_{kn}}) \cup \overline{a_{hj}}) \cup$$
$$\cup \overline{a_{ij}}(\bigcup_{h=1}^{n}(\bigcup_{k=1}^{n} a_{ik}\alpha_{kh})a_{hj} \cup \alpha_{ij}(\prod_{h=1}^{n}(\prod_{k=1}^{n}(\overline{\alpha_{ik}} \cup \overline{a_{kh}}) \cup \overline{\alpha_{hj}}) \cup$$
$$\cup \overline{\alpha_{ij}}(\bigcup_{h=1}^{n}(\bigcup_{k=1}^{n} \alpha_{ik}a_{kn})\alpha_{hj}]\} = 0,$$

where $\alpha = (\alpha_{11}, \ldots \alpha_{1n}, \alpha_{21}, \ldots \alpha_{2n}, \ldots \alpha_{n1}, \ldots \alpha_{nn})$.

Example. Let $n = 2$, the Boolean matrices in S_2 are

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \; X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}, \; a_{ij} \in \{0, 1\}$$

Statement (1) in the anti-inverse semigroup S_2 is equivalent to the system of Boolean equations $(\forall A \in S_2)(\exists X \in S_2)$

$$(7) \qquad \begin{aligned}
a_{11}x_{11} \cup a_{21}x_{11}x_{12} \cup a_{12}x_{12}x_{21} \cup a_{22}x_{21}x_{22} &= a_{11} \\
a_{11}x_{11}x_{12} \cup a_{21}x_{12} \cup a_{12}x_{11}x_{22} \cup a_{22}x_{11}x_{12} &= a_{12} \\
a_{11}x_{11}x_{21} \cup a_{21}x_{11}x_{22} \cup a_{12}x_{21} \cup a_{22}x_{21}x_{22} &= a_{21} \\
a_{11}x_{11}x_{12} \cup a_{21}x_{12}x_{22} \cup a_{12}x_{21}x_{12} \cup a_{22}x_{22} &= a_{22} \\
a_{11}x_{11} \cup a_{11}a_{21}x_{12} \cup a_{11}a_{21}x_{12} \cup a_{21}a_{22}x_{22} &= x_{11} \\
a_{11}a_{12}x_{11} \cup a_{12}x_{21} \cup a_{11}a_{22}x_{12} \cup a_{12}a_{22}x_{22} &= x_{12} \\
a_{11}a_{21}x_{11} \cup a_{11}a_{22}x_{21} \cup a_{21}x_{12} \cup a_{21}a_{22}x_{22} &= x_{21} \\
a_{21}a_{12}x_{11} \cup a_{12}a_{22}x_{21} \cup a_{21}x_{22}x_{12} \cup a_{22}x_{22} &= x_{22}
\end{aligned}$$

Corollary 1. *System (7) has a solution X for a given matrix A from S_2 if and only if*

$$(8)\qquad \begin{aligned}&(a_{11}\cup a_{12}\cup a_{21}\cup a_{22})(\overline{a_{11}}\cup a_{12}\cup a_{21}\cup a_{22})\cdot\\ &(a_{11}\cup a_{12}\cup a_{21}\cup \overline{a_{22}})(\overline{a_{11}}\cup \overline{a_{12}}\cup a_{21}\cup a_{22})\cdot\\ &(\overline{a_{11}}\cup a_{12}\cup \overline{a_{21}}\cup a_{22})(\overline{a_{11}}\cup a_{12}\cup a_{21}\cup \overline{a_{22}})\cdot\\ &(a_{11}\cup \overline{a_{12}}\cup \overline{a_{21}}\cup a_{22})(a_{11}\cup \overline{a_{12}}\cup a_{21}\cup \overline{a_{22}})\cdot\\ &(a_{11}\cup a_{12}\cup \overline{a_{21}}\cup \overline{a_{22}})(\overline{a_{11}}\cup a_{12}\cup \overline{a_{21}}\cup \overline{a_{22}})\cdot\\ &(\overline{a_{11}}\cup \overline{a_{12}}\cup a_{21}\cup \overline{a_{22}})(\overline{a_{11}}\cup \overline{a_{12}}\cup \overline{a_{21}}\cup \overline{a_{22}}) = 0\end{aligned}$$

Proof. Taking a Boolean matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

in (6) we get (8). Condition (8) is the necessary and sufficient one for a 2×2 Boolean matrix A to be an element of the anti-inverse semigroup S_2.

If we define a binary operation " $\oplus$ " on M_n

$$A \oplus B[\sum_{k=1}^{n} a_{ik}b_{ij}]\ ,\ \ i,j = 1,\ \dots ,n$$

where $a+b = \overline{a}b \cup a\overline{b}$, then $(M_n, \oplus)$ is a semugroup, $S_n^{\oplus} \subset M_n$.

Theorem 4. *A Boolean matrix A is an element of the anti-inverse semigroup $S_n^{\oplus}$ if and only if the elements of A satisfy the condition:*

$$\prod_{\alpha\in B^{n^2}} \{ \bigcup_{i,j=1}^{n} [\overline{a_{ij}} \sum_{h=1}^{n}(\sum_{k=1}^{n} a_{ik}\alpha_{kh})a_{hj} \cup a_{ij}\overline{\sum_{h=1}^{n}(\sum_{k=1}^{n} a_{ik}\alpha_{kh})a_{hj}} \cup$$
$$\cup \overline{\alpha_{ij}} \sum_{h=1}^{n}(\sum_{k=1}^{n} \alpha_{ik}a_{kh})\alpha_{hj} \cup \alpha_{ij}\overline{\sum_{k=1}^{n}(\sum_{k=1}^{n} \alpha_{ik}a_{kh})\alpha_{hj}}]\} = 0,$$

where

$$\overline{\sum_{i=1}^{n}} x_i = x_1 + \dots + x_{k-1} + \overline{x_k} + x_{k+1} + \dots + x_n\ ,\ k = 1,\ \dots ,n.$$

The proofs for Theorem 4 and Theorem 3 are similar.

References

[1] Bogdanović, S. Milić, S., V. Pavlović, V., Anti-inverse semigroups, Publ. Inst. Math. Belgrade, 24(38), 1978, p.p. 19-28

[2] Rudenau, S., Boolean functions and equations, North Holland Publishing Company, Amsterdam - New York, 1974.

REZIME

BULOVE ANTI-INVERZNE SEMIGRUPE

U ovom radu dat je potreban i dovojan uslov za Bulovu matricu da pripada Bulovoj anti-inverznoj semigrupi.

Received by the editors May 22, 1991