Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 23, 2 (1993), 211 - 219

Review of Research
Faculty of Science
Mathematics Series

ENUMERATION OF 2-FACTORS OF $P_5 \times P_n$

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Abstract

A recurrence relation for the number of 2-factors of the cartesian product $P_5 \times P_n$ is derived in the paper. By solving the recurrence relation we obtain an explicit formula for the number $f(n)$ of 2-factors in $P_5 \times P_n$ is obtained.

AMS Mathematics Subject Classification (1991): 05C70
Key words and phrases: Cartesian product, 2-factors, walks in graphs.

1. Introduction

Let P_n denote a path with n vertices, and let $f_m(n)$ be the number of 2-factors in the cartesian product $P_m \times P_n$.

Since $P_m \times P_n$ is isomorphic to $P_n \times P_m$, we may consider the vertex-set of $P_m \times P_n$ as $\{0, 1, \cdots, m-1\} \times \{0, 1, \cdots, n-1\}$ so that $P_m \times P_n$ can be represented graphically as an m-by-n grid in the usual cartesian plane. For instance , Figure 1. contains such a representation of $P_6 \times P_7$, with one of its 2-factors drawn in bold lines. It is easy to prove the following statement.

Theorem 1. *$P_m \times P_n$ has a 2-factor iff the number of vertices is even, i.e. iff at least one of the numbers m, n is even.*

From now on we shall consider only the case when at least one of m and n is even.

It is obvious that $F_1(n) = F_m(1) = 0$ for $n, m \geq 1$

Since $F_m(n) = F_n(m)$, we may take $3 \leq m \leq n$, without the loss of generality.

Consider now a labelled graph $P_m \times P_n$ and any of its 2-factors. The total number of cells of that graph is $(m-1)\cdot(n-1)$. With each cell of that graph we associate an element of the set $\{0,1\}$ in the following way : if the cell w lies in the interiors of an odd number of circuits of the given 2-factors, then the element associated with w is 1, in all other cases the associated member is 0.

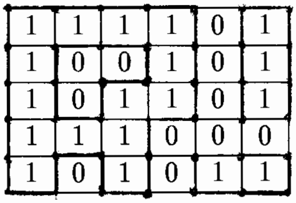

Fig.1

In that way, with each 2-factors of the labeled graph $P_m \times P_n$ we associated uniquely a binary matrix $A = [a_{i,j}]_{(m-1)\times(n-1)}$ which satisfies the following conditions:

- The first adjacency condition for two columns:

$$(1) \qquad (\forall j)(1 \leq j \leq n-2)$$

$$\neg(a_{1,j} = a_{1,j+1} = 0 \quad \vee \quad a_{m-1,j} = a_{m-1,j+1} = 0)$$

(two adjacent zeros are not allowed in the first or in the last row).

- The second adjacency condition for two columns:

$$(\forall i)(1 \leq i \leq (m-2))(\forall j)(1 \leq j \leq (n-2))$$

$$(a_{i,j}, a_{i+1,j}, a_{i,j+1}, a_{i+1,j+1}) \notin \{(0,0,0,0),(1,1,1,1),(1,0,0,1),(0,1,1,0)\} \quad (2)$$

- The first condition for the first and the last columns:

$$(3) \qquad (a_{1,1} = a_{m-1,1} = a_{1,n-1} = a_{m-1,n-1} = 1)$$

- The second condition for the first and the last columns:

$$(4) \qquad (\forall i)(1 \le i \le m-2)$$

$$\neg(a_{i,1} = a_{i+1,1} = 0 \ \lor \ a_{i,n-1} = a_{i+1,n-1} = 0)$$

Conversely, it can be proved that with each binary matrix $A = [a_{i,j}]_{(m-1)\times(n-1)}$ satisfying the conditions (1) - (4) a 2-factor of $P_m \times P_n$ can be uniquely associated. In that way a bijection is established between all 2-factors of the labelled graph $P_m \times P_n$ and all binary matrices $A = [a_{i,j}]_{(m-1)\times(n-1)}$ satisfying the conditions (1) - (4) .

We are now going to solve the equivalent problem of enumeration of such matrices. For given m, we consider graph D with the vertex set $V(D) = \{0, 1, \dots, 2^{m-1} - 1\}$ in which the two vertices p and q are adjacent iff the numbers p and q satisfy the following condition: Let $\overline{p_1 p_2 \cdots p_{m-1}}$ and $\overline{q_1 q_2 \cdots q_{m-1}}$ be the binary representations of p and q , then:

$$(5) \qquad (\forall j)(1 \le j \le n-2)\neg(p_1 = q_1 = 0 \ \lor \ p_{m-1} = q_{m-1} = 0)$$

i

$$(\forall i)(1 \le i \le m-2)(\forall j)(1 \le j \le n-2)$$

$$(6) \qquad (p_i, p_{i+1}, q_i, q_{i+1}) \notin \{(0,0,0,0),(1,1,1,1),(1,0,0,1),(0,1,1,0)\}$$

Definition 1. *A vertex* $p \in V(D)$ *is said to be the main vertex ; if its binary representation* $\overline{p_1 p_2 \cdots p_{m-1}}$ *satisfies the following conditions:*

$$(7) \qquad p_1 = p_{m-1} = 1$$

and

$$(8) \qquad (\forall i)(1 \le i \le m-1)\neg(p_i = p_{i+1} = 0)$$

In this way the problem of enumeration of all binary matrices $A = [a_{i,j}]_{(m-1)\times(n-1)}$ satisfying the conditions (1) - (4) is reduced to the problem of enumeration of all walks of the length $n-2$ in D with the initial and final vertices in the set of main vertices.

Let $m = 5$. The adjacency matrix of the associated digraph D is

$$\begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&1\\
0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&1\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1\\
0&0&0&0&0&0&0&0&1&1&1&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0\\
0&0&0&1&0&0&0&0&0&0&0&1&0&1&0&1\\
0&0&0&1&0&0&0&0&0&0&0&1&1&1&0&1\\
0&0&0&1&0&0&0&0&0&0&0&1&0&0&0&1\\
1&1&1&0&0&0&0&0&1&1&1&0&0&0&1&0\\
0&1&0&0&0&1&0&1&0&1&0&0&0&0&0&0\\
1&1&0&0&1&1&0&1&1&1&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&1&0&0&0&0\\
1&1&1&0&1&1&0&0&1&1&1&0&0&0&0&0
\end{bmatrix}$$

The set of main vertices is $\{11, 13, 15\}$.

If we denote by $f_i(k)$ the number of walks of the length k having the initial vertex i and the final vertex is a main vertex, then for $m = 5$, we have:

$$f_0(k) = f_{11}(k-1) + f_{13}(k-1) + f_{15}(k-1)$$

$$f_1(k) = f_{11}(k-1) + f_{12}(k-1) + f_{13}(k-1) + f_{15}(k-1)$$

$$f_2(k) = f_{11}(k-1) + f_{15}(k-1)$$

$$f_3(k) = f_8(k-1) + f_9(k-1) + f_{10}(k-1) + f_{14}(k-1)$$

$$f_4(k) = f_{13}(k-1) + f_{15}(k-1)$$

$$f_5(k) = f_{12}(k-1) + f_{13}(k-1) + f_{15}(k-1)$$

$$f_6(k) = 0$$
$$f_7(k) = f_{12}(k-1) + f_{13}(k-1)$$
$$f_8(k) = f_3(k-1) + f_{11}(k-1) + f_{13}(k-1) + f_{15}(k-1)$$
$$f_9(k) = f_3(k-1) + f_{11}(k-1) + f_{12}(k-1) + f_{13}(k-1) + f_{15}(k-1)$$
$$f_{10}(k) = f_3(k-1) + f_{11}(k-1) + f_{15}(k-1)$$
$$f_{11}(k) = f_0(k-1)+f_1(k-1)+f_2(k-1)+f_8(k-1)+f_9(k-1)+f_{10}(k-1)+f_{14}(k-1)$$
$$f_{12}(k) = f_1(k-1) + f_5(k-1) + f_7(k-1) + f_9(k-1)$$
$$f_{13}(k) = f_0(k-1)+f_1(k-1)+f_4(k-1)+f_5(k-1)+f_7(k-1)+f_8(k-1)+f_9(k-1)$$
$$f_{14}(k) = f_4(k-1) + f_{11}(k-1)$$
$$f_{15}(k) = f_0(k-1)+f_1(k-1)+f_2(k-1)+f_4(k-1)+f_5(k-1)+f_8(k-1)+f_9(k-1)+f_{10}(k-1$$
$$f(k) = f_{11}(k-2) + f_{13}(k-2) + f_{15}(k-2)$$

Now, it is easy to see that:

$$f_1(k) = f_8(k)$$
$$f_2(k) = f_4(k)$$
$$f_3(k) = f_{12}(k)$$
$$f_5(k) = f_{10}(k)$$
$$f_7(k) = f_{14}(k)$$
$$f_{11}(k) = f_{13}(k)$$

and the system can be reduced to:

(9) $$f_0(k) = 2 \cdot f_{11}(k-1) + f_{15}(k-1)$$

(10) $$f_1(k) = f_3(k-1) + 2 \cdot f_{11}(k-1) + f_{15}(k-1)$$

(11) $$f_2(k) = f_{11}(k-1) + f_{15}(k-1)$$

$$f_3(k) = f_1(k-1) + f_5(k-1) + f_7(k-1) + f_9(k-1)$$

(12) $$f_5(k) = f_3(k-1) + f_{11}(k-1) + f_{15}(k-1)$$

$$f_7(k) = f_3(k-1) + f_{11}(k-1)$$

(13) $$f_9(k) = 2 \cdot f_3(k-1) + 2 \cdot f_{11}(k-1) + f_{15}(k-1)$$

(14) $$f_{11}(k) = f_0(k-1) + 2 \cdot f_1(k-1) + f_2(k-1) + f_5(k-1) +$$

$$+f_7(k-1)+f_9(k-1)$$

(15) $$f_{15}(k)=f_0(k-1)+2\cdot f_1(k-1)+2\cdot f_2(k-1)+$$

$$+2\cdot f_5(k-1)+f_9(k-1)$$

(16) $$f(k)=2\cdot f_{11}(k-2)+f_{15}(k-2)$$

Using (9) and (16) in (10) - (15), the system can be reduced to:

(17) $$f_1(k)=f_3(k-1)+f(k+1)$$

(18) $$f_2(k)=f(k+1)-f_{11}(k-1)$$

(19) $$f_3(k)=f_1(k-1)+f_5(k-1)+f_7(k-1)+f_9(k-1)$$

(20) $$f_5(k)=f_3(k-1)-f_{11}(k-1)+f(k+1)$$

(21) $$f_7(k)=f_3(k-1)+f_{11}(k-1)$$

(22) $$f_9(k)=2\cdot f_3(k-1)+f(k+1)$$

(23) $$f_{11}(k)=2\cdot f_1(k-1)+f_2(k-1)+f_5(k-1)+$$

$$+f_7(k-1)+f_9(k-1)+f(k)$$

(24) $$f_{15}(k)=2\cdot f_1(k-1)+2\cdot f_2(k-1)+$$

$$+2\cdot f_5(k-1)+f_9(k-1)+f(k)$$

$$f(k)=2\cdot f_{11}(k-2)+f_{15}(k-2)$$

Substituing (17) , (18) , (20) , (21) and (22) into (19),(20) and (21) the system is transformed into:

(25) $$f_3(k)=5\cdot f_3(k-2)+3\cdot f(k)$$

(26) $$f_{11}(k)=6\cdot f_3(k-2)-f_{11}(k-2)+6\cdot f(k)$$

(27) $$f_{15}(k)=6\cdot f_3(k-2)-f_{11}(k-2)+8\cdot f(k)$$

(28) $$f(k)=2\cdot f_{11}(k-2)+f_{15}(k-2)$$

It follows from (26)

$$6 \cdot f_3(k-2) = f_{11}(k) + f_{11}(k-2) - 6 \cdot f(k)$$

Applying this , we obtain from (27):

$$f_{15}(k) = f_{11}(k) - 3 \cdot f_{11}(k-2) + 2 \cdot f(k) \tag{29}$$

and from (25) (multiplying by 6):

$$5 \cdot f_{11}(k-2) + 4 \cdot f_{11}(k) - f_{11}(k+2) = 12 \cdot f(k) - 6 \cdot f(k+2) \tag{30}$$

Putting (29) into (28) we obtain:

$$3 \cdot f_{11}(k-2) - 3 \cdot f_{11}(k-4) = f(k) - 2 \cdot f(k-2) \tag{31}$$

From (31) , with $(k+2)$ instead of k , we have:

$$3 \cdot f_{11}(k) - 3 \cdot f_{11}(k-2) = f(k+2) - 2 \cdot f(k) \tag{32}$$

From (30), after multiplication by 3, we obtain:

$$15 \cdot f_{11}(k-2) + 12 \cdot f_{11}(k) - 3 \cdot f_{11}(k+2) = \\ = 36 \cdot f(k) - 18 \cdot f(k+2) \tag{33}$$

and taking k instead of $(k-2)$ we have:

$$15 \cdot f_{11}(k-4) + 12 \cdot f_{11}(k-2) - 3 \cdot f_{11}(k) = \\ = 36 \cdot f(k-2) - 18 \cdot f(k) \tag{34}$$

If we subtract (34) from (33), taking into account (32) we obtain:

$$f(k+4) - 24 \cdot f(k+2) + 57 \cdot f(k) - 26 \cdot f(k-2) = 0$$

So, the following statement is proved:

Theorem 2. *Let for* $n \geq 1$, $F(n) = f(2 \cdot n)$. *The number* $F(n)$ *of 2-factors of* $P_5 \times P_{2n}$ *satisfies the recurrence relation:*

$$F(n) = 24 \cdot F(n-1) - 57 \cdot F(n-2) + 26 \cdot F(n-3)$$

for $n \geq 4$, *with the initial conditions:* $F(1) = 3$, $F(2) = 54$, $f(3) = 1140$.

Remark. If we define $F(0) = \frac{15}{26}$, then the recurrence relacion $F(n) = 24 \cdot F(n-1) - 57 \cdot F(n-2) + 26 \cdot F(n-3)$ will be satisfied, for $n \geq 3$, with the initial condition: $F(0) = \frac{15}{16}$, $F(1) = 3$, F(2) = 54 .

Theorem 3.

$$F(n) = \frac{1}{3} \cdot 2^{n-1} + \frac{2 \cdot (4-\sqrt{3})}{39} \cdot (11 + 6 \cdot \sqrt{3})^n + \frac{2 \cdot (4+\sqrt{3})}{39} \cdot (11 - 6 \cdot \sqrt{3})^n,$$

for $n \geq 1$.

Proof.

The roots of the characteristic equation

$$x^3 - 24 \cdot x^2 + 57 \cdot x - 26 = 0$$

are: $x_1 = 2$, $x_2 = 11 + 6 \cdot \sqrt{3}$, $x_3 = 11 - 6 \cdot \sqrt{3}$.
So, the general solution of the recurrence relation $F(n) = 24 \cdot F(n-1) - 57 \cdot F(n-2) + 26 \cdot F(n-3)$ is

$$F(n) = A \cdot 2^n + B \cdot (11 + 6\sqrt{3})^n + C \cdot (11 - 6 \cdot \sqrt{3})^n \quad .$$

The constants A, B, C are determined using the initial conditions:
$A = \frac{1}{6}$, $B = \frac{2 \cdot (4-\sqrt{3})}{39}$, $C = \frac{2 \cdot (4+\sqrt{3})}{39}$.
Hence follows the above statement.

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REZIME

PREBROJAVANJE 2-FAKTORA GRAFA $P_5 \times P_n$

U radu je izvedena rekurentna relacija za broj 2-faktora Dekartovog proizvoda grafa $P_5 \times P_n$. Rešavanjem ove rekurentne relacije dobijena je eksplicitna formula za broj 2-faktora $f(n)$ grafa $P_5 \times P_n$.

Received by the editors November 20, 1991