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ON A MULTIPLICATIVE FUNCTION CONNECTED WITH THE NUMBER OF DIRECT FACTORS OF A FINITE ABELIAN GROUP

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Abstract

Let $T(x) = \sum \tau(G)$, where $\tau(G)$ denotes of direct factors of an Abelian group G. It is know that $T(x) = \sum_{n\le x} t(n)$, where $t(n)$ is a multiplicative function such that $\sum_{n=1}^{\infty} t(n)n^{-s} = \zeta^2(s)\zeta^2(2s)\zeta^2(3s)\ldots$ ($Re\ s > 1$), and $\zeta(s)$ as usual denotes the Riemann zata-function. The aim of thi note is to investigate some asymptotic formulas for the summatory functions of $t^k(n)$.

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Let $T(x) = \sum \tau(G)$, where $\tau(G)$ denotes the number of direct factors of an Abelian group G, and summation is extended over all Abelian groups whose orders do not exceed x. It is known (see E. Cohen [1] or E. Krätzel [6]) that

$$T(x) = \sum_{n\le x} t(n),$$

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where $t(n)$ is a multiplicative function such that

$$\sum_{n=1}^{\infty} t(n)n^{-s} = \zeta^2(s)\zeta^2(2s)\zeta^2(3s)\ldots \qquad (Re\ s > 1), \tag{1}$$

and $\zeta(s)$ as usual denotes the Riemann zeta-function. If $a(n)$ denotes the number of non-isomorphic Abelian groups with n elements, then it is well-known (see Ch. 14 of A. Ivić [3] or Ch. 7 of E. Krätzel [7]) that

$$\sum_{n=1}^{\infty} a(n)n^{-s} = \zeta(s)\zeta(2s)\zeta(3s)\ldots \qquad (Re\ s > 1). \tag{2}$$

Historically, the summatory function of $a(n)$ was first investigated by Erdős-Szekeres [2] in 1935, and from that time much research was done on this subject (see [3] or [7] for some of the references). One has $a(p^\alpha) = P(\alpha)$ for any prime p and integer $\alpha \geq 1$, where $P(\alpha)$ is the number of (unresticted) partitions of α. From (1) and (2) it follows that

$$\sum_{n=1}^{\infty} t(n)n^{-s} = (\sum_{n=1}^{\infty} a(n)n^{-s})^2 \qquad (Re\ s > 1),$$

hence for any integer $n \geq 1$

$$t(n) = \sum_{d|n} a(d)a(\frac{n}{d}). \tag{3}$$

Thus for any prime p and integer $\alpha \geq 1$

$$t(p^\alpha) = 2P(\alpha) + \sum_{j=1}^{\alpha-1} P(j)P(\alpha - j). \tag{4}$$

In particular, since $P(1) = 1$, $P(2) = 2$, $P(3) = 3$, $P(4) = 5$, we have

$$t(p) = 2,\ t(p^2) = 5,\ t(p^3) = 10,\ t(p^4) = 20, \tag{5}$$

and in view of $a(n) \ll_\epsilon n^\epsilon$ (ϵ denotes arbitrarily small positive constants, not necessarily the same ones at each occurrence) (3) yields

$$t(n) \ll_\epsilon n^\epsilon. \tag{6}$$

Here, as usual, the symbol $f(x) \ll g(x)$ (same as $f(x) = O(g(x))$) means that $|f(x)| \leq Cg(x)$ for some $C > 0$, $x \geq x_0$ if $g(x) > 0$. The symbol $f(x) \ll_\epsilon g(x)$ means that the constant C may depend on ϵ.

The function

$$\begin{aligned}\Delta_1(x) \quad &:= \quad T(x) - \sum_{j=1}^{5} \operatorname*{Res}_{s=1/j} \prod_{k=1}^{\infty} \zeta^2(ks)x^s s^{-1} \\ &= \quad \sum_{mn\leq x} a(m)a(n) - \sum_{j=1}^{5}(D_j \log x + E_j)x^{1/j},\end{aligned}$$

where D_j, E_j are suitable constants which may be explicitly evaluated, and $|Delta_1(x)$ may be thought of as the error term in the asymptotic formula for $T(x)$. E. Krätzel [6] proved that

$$\Delta_1(x) \ll x^{5/12} \log^4 x,$$

and this result was improved by Menzer-Seibold [8] to

$$\Delta_1(x) \ll_\epsilon x^{45/109+\epsilon}, \quad \frac{45}{109} = 0.412844\ldots .$$

Averages of $\Delta_1(x)$ were considered by the author [5], who proved

$$\int_1^X \Delta_1(x)dx \ll_\epsilon X^{7/6+\epsilon} \tag{7}$$

and

$$\int_1^X \Delta_1^2(x)dx = \Omega(X^{3/2} \log^4 X), \quad \int_1^X \Delta_1^2(x)dx \ll_\epsilon X^{8/5+\epsilon}, \tag{8}$$

where as usual $f(x) = \Omega(g(x))$ means that $\lim_{x\to\infty} f(x)/g(x) = 0$ does not hold. The bound in (7) makes it clear why it is appropriate to have summation from $j = 1$ to $j = 5$ in the definition of $\Delta_1(x)$. The Ω - result in (8) makes the conjecture

$$\Delta_1(x) \ll_\epsilon x^{1/4+\epsilon}$$

plausible, although if true, this bound will be very hard to prove.

The aim of this note is to investigate some asymptotic formulas for the summatory functions of $t^k(n)$. Let $P_k(y)$ denote a generic polynomial of degree k in y whose coefficients may be explicitly evaluated. Then we have

Theorem 1. *For any given $\epsilon > 0$*

$$\sum_{n\le x} t^2(n) = xP_3(\log x) + O(x^{1/2}\log^{20} x), \tag{9}$$

$$\sum_{n\le x} t(n^2) = xP_4(\log x) + O_\epsilon(x^{11/20+\epsilon}), \tag{10}$$

$$\sum_{n\le x} t^3(n) = xP_7(\log x) + O_\epsilon(x^{5/8+\epsilon}), \tag{11}$$

$$\sum_{n\le x} t^4(n) = xP_{15}(\log x) + O(x^{77/100}). \tag{12}$$

Proof. By using the multiplicativity of $t(n)$, (5) and (6), we have, for $Re\ s > 1$,

$$\sum_{n=1}^{\infty} t^2(n)n^{-s} = \prod_p (1 + \sum_{j=1}^{\infty} t^2(p^j)p^{-js})$$

(13)

$$= \prod_p (1 + 4p^{-s} + 25p^{-2s} + 100p^{-3s} + \ldots)$$

$$= \zeta^4(s)\prod_p (1-4p^{-s}+6p^{-2s}-4p^{-3s}+p^{-4s})(1+4p^{-s}+25p^{-2s}+100p^{-3s}+\ldots)$$

$$= \zeta^4(s)\prod_p (1 + 15p^{-2s} + \sum_{j=3}^{\infty} C_j p^{-js}) = \zeta^4(s)\zeta^{15}(2s)A(s),$$

where the C_j's are suitable constants, and $A(s)$ represents a Dirichlet series which converges absolutely for $Re\ s > 1/3$. In a similar way it may be seen that

$$\sum_{n=1}^{\infty} t(n^2)n^{-s} = \zeta^5(s)B(s), \tag{14}$$

$$\sum_{n=1}^{\infty} t^3(n)n^{-s} = \zeta^8(s)C(s), \tag{15}$$

$$\sum_{n=1}^{\infty} t^4(n)n^{-s} = \zeta^{16}(s)D(s), \tag{16}$$

where $B(s), C(s), D(s)$ represent Dirichlet series all of which converge absolutely for $Re\ s > 1/2$.

To prove (9) we use the representation (13) and the truncated Perron's inversion formula for Dirichlet series (see the Appendix of [3]). Thus for $x^\epsilon \ll T \ll x$ we have

$$\begin{aligned}\sum_{n\le x} t^2(n) &= \frac{1}{2\pi i}\int_{1+\epsilon-iT}^{1+\epsilon+iT} \zeta^4(s)\zeta^{15}(2s)A(s)x^s s^{-1}ds + O_\epsilon(x^{1+\epsilon}T^{-1})\\ &= xP_3(\log x) + I_1 + I_2 + I_3 + O_\epsilon(x^{1+\epsilon}T^{-1})\end{aligned}$$

by the residue theorem, where we set

$$I_1 = \frac{1}{2\pi i}\int_{1/2-iT}^{1/2+iT} \zeta^4(s)\zeta^{15}(2s)A(s)x^s s^{-1}ds,$$

$$I_2 = \frac{1}{2\pi i}\int_{1/2+iT}^{1+\epsilon+iT} \zeta^4(s)\zeta^{15}(2s)A(s)x^s s^{-1}ds,$$

$$I_3 = \frac{1}{2\pi i}\int_{1+\epsilon-iT}^{1/2-iT} \zeta^4(s)\zeta^{15}(2s)A(s)x^s s^{-1}ds.$$

By using the estimate (all the necessary results on $\zeta(s)$ are to be found in [3])

$$\zeta(\sigma+it) \ll (|t|^{(1-\sigma)/3}+1)\log|t| \qquad (1/2\le\sigma\le 2)$$

we obtain

$$I_2+I_3 \ll \int_{1/2}^{1+\epsilon} |\zeta(\sigma+iT)|^4\log^{15}T\frac{x^\sigma d\sigma}{T} \ll \frac{\log^{19}T}{T}(x^{1/2}T^{2/3}+x^{1+\epsilon}).$$

From the weak estimate

$$\int_1^T |\zeta(\tfrac{1}{2}+it)|^4 dt \ll T\log^4 T$$

and integration by parts it follows that

$$I_1 \ll x^{1/2}\log^{20}T.$$

Hence

$$\sum_{n\le x} t^2(n) = xP_3(\log x) + O_\epsilon(x^{1+\epsilon}T^{-1}) + O(x^{1/2}\log^{20}T)$$

and (9) follows with $T = x^{1/2+\epsilon}$. It may be remarked that the log-power in (9) may be improved by using the bound $\zeta(1+it) \ll \log^{2/3}|t|$, which is the sharpest one of its kind.

The remaining asymptotic formulas in Theorem 1 are proved analogously, if one notes that by Theorem 8.4 of [3]

$$\int_1^T |\zeta(\frac{11}{20}+it)|^5 dt \ll_\epsilon T^{1+\epsilon}, \tag{17}$$

$$\int_1^T |\zeta(\frac{5}{8}+it)|^8 dt \ll_\epsilon T^{1+\epsilon}, \tag{18}$$

$$\int_1^T |\zeta(\sigma_0+it)|^{16} dt \ll_\epsilon T^{1+\epsilon}, \qquad \sigma_0 = 0.769229\ldots . \tag{19}$$

Namely the product representations in (14)-(16) are dominated by $\zeta^5(s)$, $\zeta^8(s)$, $\zeta^{16}(s)$, respectively. Therefore for the summatory functions of $t(n^2)$, $t^3(n)$ and $t^4(n)$ we use again Perron's formula and shift the segment of integration to $Re\ s = 11/20,\ 5/8,\ \sigma_0$ respectively, in view of (17)-(19). Hence we obtain (10)-(12), the last formula because $\sigma_0 < 77/100$. In the same way we obtain the asymptotic formula

$$\sum_{n\le x} t^k(n) = xP_{2^k-1}(\log x) + O_{\epsilon,k}(x^{c_k+\epsilon})$$

for any integer $k \ge 1$, where $c_k(<1)$ is a suitable constant. However, this constant clearly depends on results on power moments of $\zeta(s)$, and for this reason its explicit form for general k would be complicated.

We also note that the product representation (13) gives us reason to believe that very likely a sharper asymptotic formula than (9) holds for the summatory function of $t^2(n)$. Namely, the factor $\zeta^{15}(2s)$ hints at the existence of a second main term and it makes sence to define

$$\begin{aligned}\Delta_2(x) :&= \sum_{n\le x} t^2(n) - \sum_{j=1}^{2} \operatorname*{Res}_{s=1/j}\ \zeta^4(s)\zeta^{15}(2s)A(s)x^s s^{-1} \\ &= \sum_{n\le x} t^2(n) - xP_3(\log x) - x^{1/2}P_{14}(\log x)\end{aligned}$$

and to expect that

$$\Delta_2(x) = o(x^{1/2}) \qquad (x\to\infty). \tag{20}$$

The function $\Delta_2(x)$ is the analogue of the function $\Delta_1(x)$ in the problem of $\sum_{n\le x} t(n)$. I conjecture that for some $A>0$ and $B\ge 0$

$$\int_1^X \Delta_2^2(x)dx \sim AX^{7/4}\log^B X \qquad (X\to\infty) \tag{21}$$

and that even more than (20) is true, namely

$$\Delta_2(x) \ll_\epsilon x^{3/8+\epsilon}. \tag{22}$$

Both (21) and (22) appear very hard to prove. However, in the mean square sense (20) is certainly true. This, and even more, is contained in

Theorem 2. *We have*

$$\int_1^X \Delta_2^2(x)dx = \Omega(X^{7/4}), \tag{23}$$

$$\int_1^X \Delta_2^2(x)dx \ll X^{49/25}. \tag{24}$$

Moreover, if $\zeta(3/4+it)\ll_\epsilon |t|^\epsilon$ *holds, then*

$$\int_1^X \Delta_2^2(x)dx \ll_\epsilon X^{7/4+\epsilon}. \tag{25}$$

Proof. The significance of (25) is that it shows, at least conditionally, that the true order of the mean square integral of $\Delta_2(x)$ is $X^{7/4+o(1)}$ as $X\to\infty$, and hence supports the conjectural bound (22).

The Ω -result (23) follows from Theorem 3 of [4]. In our case this result may be applied with

$$a_1=a_2=a_3=a_4=1,\ a_5=2,\ r=4,$$
$$g=\frac{r-1}{2(a_1+\ldots+a_r)}=\frac{3}{8},\quad a_rg_r=\frac{3}{8}<\frac{1}{2}.$$

Thus we have $A=0$, and (23) follows.

We pass now to the proof of the upper bounds in (24) and (25). We may proceed analogously as in the proof of Theorem 2 of [5] or note that, by the Perron inversion formula,

$$\Delta_2(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\zeta^4(s)\zeta^{15}(2s)A(s)x^s s^{-1}\,ds$$

for some $c < 1/2$, but sufficiently close to $1/2$. Thus $\zeta^4(s)\zeta^{15}(2s)A(s)$ is the Mellin transform of $\Delta_2(1/x)$, and by Parseval's identity for Mellin transforms we obtain from the above relation

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |\zeta^4(c+it)\zeta^{15}(2c+2it)A(c+it)(c+it)^{-1}|^2 dt = \int_0^{\infty} \Delta_2^2(\frac{1}{x})x^{2c-1}dx$$

$$= \int_0^{\infty} \Delta_2^2(x)x^{-2c-1}dx. \tag{26}$$

From (26) it follows that if, for some $3/8 \le \sigma_1 \le 1/2$ and $\delta > 0$,

$$\int_T^{2T} |\zeta(\sigma_1+it)|^8|\zeta(2\sigma_1+2it)|^{30}dt \ll T^{2-\delta} \tag{27}$$

holds, then

$$\int_1^X \Delta_2^2(x)dx \ll X^{1+2\sigma_1}. \tag{28}$$

Suppose first $\sigma_1 = \frac{3}{8}$. Then if $\zeta(\frac{3}{4}+it) \ll_\epsilon |t|^\epsilon$ (this conjecture follows e.g. from the Lindelöf hypothesis, which in turn follows from the Riemann hypothesis), we have that the left-hand side of (27) is

$$\ll_\epsilon T^\epsilon \int_T^{2T} |\zeta(\frac{3}{8}+it)|^8 dt \ll_\epsilon T^{1+\epsilon}\int_T^{2T} |\zeta(\frac{5}{8}+it)|^8 dt \ll_\epsilon T^{2+2\epsilon},$$

where we used (18) and the functional equation

$$\zeta(s) = \chi(s)\zeta(1-s), \quad \chi(s) \asymp |t|^{1/2-\sigma}.$$

Hence if instead of $\sigma_1 = \frac{3}{8}$ we take $\sigma_1 = \frac{3}{8}+\epsilon_1$, then because of the functional equation and properties of power moments of $\zeta(s)$ we shall obtain (27). This proves (25).

To prove (24) note that

$$\int_T^{2T} |\zeta(\sigma_1+it)|^8|\zeta(2\sigma_1+2it)|^{30}dt$$

$$\ll T^{4-8\sigma_1}\int_T^{2T} |\zeta(1-\sigma_1+it)|^8|\zeta(2\sigma_1+2it)|^{30}dt$$

$$\ll T^{4-8\sigma_1} \max_{T\le t\le 2T}|\zeta(1-\sigma_1+it)|^4|\zeta(2\sigma_1+2it)|^{30}\int_T^{2T} |\zeta(1-\sigma_1+it)|^4 dt.$$

The last integral above is trivialy $\ll T\log^4 T$ if $\sigma \le 1/2$. From the bounds

$$\zeta(\sigma+it) \ll t^{c(1-\sigma)}\log t \qquad (\frac{1}{2} \le \sigma \le 1),$$

$$\zeta(\sigma+it) \ll t^{d(1-\sigma)}\log t \qquad (\frac{28}{31} \le \sigma \le 1)$$

with $c < 1/3$ and $d < 1/6$, we infer with $\sigma_1 = 12/25$ that the integral in (27) is

$$\ll_\epsilon T^{5-8\sigma_1+\epsilon+4c\sigma_1+30d(1-2\sigma_1)} \ll T^{2-\delta}$$

for sufficiently small ϵ and suitable $\delta = \delta(c,d,\epsilon) > 0$, since

$$5 - 8\sigma_1 + 4c\sigma_1 + 30d(1-2\sigma_1) < 5 - 8\sigma_1 + \frac{4}{3}\sigma_1 + 5(1-2\sigma_1) = 2$$

for $\sigma_1 = \frac{12}{25}$. Thus (27) holds with $\sigma_1 = \frac{12}{25}$ and (24) is proved. By more careful considerations the exponent in (24) could be somewhat reduced, but the existing methods and known results on power moments of $\zeta(s)$ do not seem sufficiently strong to yield an unconditional proof of (25).

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REZIME

O MULTIPLIKATIVNIM FUNKCIJAMA POVEZANIM SA BROJEM DIREKTNIH FAKTORA KONAČNE ABELOVE GRUPE

Neka je $T(x) = \sum \tau(G)$, gde $\tau(G)$ označava broj direktnih faktora Abelove grupe G i sumiranje je izvršeno nad svim Abelovim grupama čiji redovi ne prelaze x. Poznato je da je (videti E. Cohen [1] ili E. Krätzel [6]) tako da je

$$T(x) = \sum_{n \leq x} t(n),$$

gde je $t(n)$ multiplikativna funkcija takva da je

$$\sum_{n=1}^{\infty} t(n) n^{-s} = \zeta^2(s)\zeta^2(2s)\zeta^2(3s)\ldots \quad (Re\, s > 1).$$

Cilj rada je ispitivanje nekih asimptotskih formula za sumacione funkcije $t^k(n)$.

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