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ON SEMI-INNER PRODUCT SPACES OF TYPE (p)

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Abstract

In this paper we give a simple proof of the fact that every normed vector space can be made into a semi-inner product space of type (p), introduced by B. Nath as a generalization of Lumer's semi-inner product space. Introducing the homogenity property of the semi-inner product of type (p) we prove the Riesz Representation theorem.

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1. Introduction

G. Lumer [4] introduced the notion of semi-inner product on arbitrary Banach space, with the aim to transfer Hilbert space techniques to the theory of Banach spaces (see also [1], [2], [3], [6]). B. Nath [5] has generalized this notion to the semi-inner product of type (p) (originaly: generalized semi-inner product). We shall give in this paper a simple proof of B. Nath result from [5] that every normed vector space can be made into a semi-inner product space of type (p) and conversely each semi-inner product space of type (p) is a normed space. Imposing further restrictions we shall prove the Riesz Representation theorem on linear continuous functionals.

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2. Semi-inner product of type (p)

Let X be a vector space over the field K of real or complex numbers.

Definition 1. *If a functional* $[x,y],[\cdot,\cdot]: X \times X \to K$ *satisfies the following conditions*

$$
\begin{aligned}
&(1) \quad [x+y,z] = [x,z] + [y,z], \quad x,y,z \in X,\\
&(2) \quad [\lambda x, y] = \lambda [x,y], \quad \lambda \in K \text{ and } x,y \in X,\\
&(3) \quad [x,x] > 0 \text{ for } x \neq 0,\\
&(4) \quad |\,[x,y]\,| \leq [x,x]^{\frac{1}{p}} [y,y]^{\frac{p-1}{p}}, \; 1 < p < \infty,
\end{aligned}
$$

then, we say that $[x,y]$ *is a semi-inner product of type (p).*

A vector space X, together with a semi-inner product of type (p) defined on it, will be called a semi-inner product space of type (p) (s.i.p.s.(p)).

Particulary a s.i.p.s.(2) becomes Lumer's [4] semi-inner product space. Krein spaces and particularly Pontrjagin spaces are s.i.p.s. (2) (see [1], [2]).B. Nath [5] has proved the following.

Theorem 1. *A s.i.p.s.(p) becomes a normed space with the norm* $[x,x]^{1/p}$, $1 < p < +\infty$. *Every normed vector space can be made into a s.i.p.s.(p).*

We shall give a shorter and simpler proof then that in [5].
Proof. First, we shall prove that a s.i.p.s.(p) is a normed vector space with the norm $\|x\| := [x,x]^{1/p}$.

a) We have

$$
\begin{aligned}
\|x+y\|^p &= [x+y,x+y] = [x,x+y] + [y,x+y] \leq \\
&\leq [x,x]^{1/p}[x+y,x+y]^{\frac{p-1}{p}} + [y,y]^{1/p}[x+y,x+y]^{\frac{p-1}{p}} = \\
&= \|x\|\,\|x+y\|^{p-1} + \|y\|\,\|x+y\|^{p-1} = (\|x\| + \|y\|)\|x+y\|^{p-1}
\end{aligned}
$$

Dividing by $\|x+y\|^{p-1}$, for $x+y \neq 0$ the proceding inequality we obtain

$$\|x+y\| \leq \|x\| + \|y\|.$$

Obviously, the proceding inequality is true also for $x + y = 0$.

b) We have for $\lambda \in K$

$$\|\lambda x\|^p = [\lambda x, \lambda x] = \lambda[x, \lambda x] \leq$$
$$\leq |\lambda| \, [x,x]^{1/p}[\lambda x, \lambda x]^{\frac{p-1}{p}} = |\lambda| \, \|x\| \, \|\lambda x\|^{p-1}$$

Dividing by $\|\lambda x\|^{p-1}$ for $\lambda \neq 0$ and $x \neq 0$, the proceding inequality we obtain

$$\|\lambda x\| \leq |\lambda| \, \|x\|.$$

Particulary we have for $\lambda \neq 0$

$$\|x\| = \|\frac{1}{\lambda}\lambda x\| \leq \frac{1}{|\lambda|}\|\lambda x\|, \; i.e. \; |\lambda| \, \|x\| \leq \|\lambda x\|.$$

Hence by both inequalites $\|\lambda x\| = |\lambda| \, \|x\|$. The proceding inequality is obviously true also for $\lambda = 0$ or $x = 0$.

c) By (2) and (4) we have

$$\|x\| = 0 \text{ iff } x = 0.$$

Now, suppose that X is an arbitrary normed space with the norm $\|\cdot\|\cdot$. Let y be an arbitrary element from X, which we represent in the form $y = \alpha y_0$, where $\alpha = \|y\|$ and $y_0 = y \cdot \|y\|^{-1}$. Since $\|y_0\| = 1$, there exists at least one (we choose excactly one) $y_0' \in X'$, such that $y_0'(y_0) = 1$ and $\|y_0'\| = 1$ by as a consequence of Hahn-Banach theorem.

We correspond to $y = \alpha y_0$ and $p > 1$ the functional $y' = \alpha^{p-1} y_0$. We claim that

$$[x, y] := y'(x) = \alpha^{p-1} y_0'(x)$$

is a semi-inner product such that $\|x\| = [x,x]^{1/p}$. We can easy check the properties (1) - (4):

(1) $[x + y, z] = z'(x + y) = z'(x) + z'(y) = [x, z] + [y, z]$,

(2) $[\lambda x, y] = y'(\lambda x) = \lambda y'(x) = \lambda[x, y]$ for $\lambda \in K$,

(3) Let$x \neq 0$.Then for $x = \alpha x_0$, $\alpha = \|x\|$ and $x_0 = x \cdot \|x\|^{-1}$ we have $\alpha \neq 0$and$[x, x] = [\alpha x_0, \alpha x_0] = (\alpha x_0)' \, (\alpha x_0) = \alpha^{p-1} x_0'(\alpha x_0) = \alpha^p \neq 0$, where x_0' is the functional with the property $x_0'(x_0) = 1$.

(4) Let $x, y \in X$.We have $x = \alpha_0$, where $\alpha = \|x\|$ and $x_0 = x \cdot \|x\|^{-1}$, and $y = \beta y_0$, where $\beta = \|y\|$ and $y_0 = y \cdot \|y\|^{-1}$.Then we have

$$| [x,y] | = | [\alpha x_0, \beta y_0] | = |(\beta y_0)'(\alpha x_0)| = |\beta^{p-1} y_0'(\alpha x_0)| = \beta^{p-1} \cdot \alpha \cdot |y_0'(x_0)| \leq$$

$$\leq \beta^{p-1} \cdot \alpha \|y_0'\| \, \|x_0\| = \beta^{p-1} \cdot \alpha = [x,x]^{1/p} \cdot [y,y]^{\frac{p-1}{p}}$$

(for the last equality see (3)).

Since $[x,x] = \alpha^p = \|x\|^p$, we obtain $\|x\| = [x,x]^{1/p}$.

3. Riesz representation

Definition 2. *A s.i.p.s.(p) X, for $p > 1$, is said to be continuous if for every $x, y \in X$ such that $\|x\| = \|y\| = 1$*

$$Re[y, x + \lambda y] \to Re[y, x]$$

for all real $\lambda \to 0$.

Definition 3. *For $x, y \in X$, x is normal to y and y is transversal to x if $[x,y] = 0$.*

With a minor changes we can prove in an analogous way as Theorem 2. in [3] the following.

Theorem 2. *In a continuous s.i.p.s.(p) an element x is normal to y if and only if*

$$\|x + \lambda y\| \geq \|x\| \text{ for all } \lambda \in K.$$

Since the proof of Lemma 4. from [3] is based only on a property of uniformly convex Banach space we have immidiately the following generalization of Lemma 4 from [3].

Lemma 1. *In a continuous s.i.p.s.(p) which is uniformly convex and complete there exists a nonzero vector normal to every proper closed vector subspace.*

Definition 4. *A normed vector space is strictly convex if the condition $\|x\| + \|y\| = \|x + y\|$ for $x, y \neq 0$ implies $y = \lambda x$ for some real $\lambda > 0$.*

We have the following generalization of the part ol Lemma 5. from [3].

Lemma 2. *If an s.i.p.s.(p), for* $p > 1$, *is strictly convex, the condition* $[x,y] = \|x\| \cdot \|y\|^{p-1}$ *for* $x, y \neq 0$ *implies* $y = \lambda x$ *for some real* $\lambda > 0$.

Proof. Suppose that for $x \neq 0$ and $y \neq 0$ holds $[x,y] = \|x\|\,\|y\|^{p-1}$. Then we have

$$\|x\| + \|y\| \geq \|x+y\| = \frac{\|x+y\|\,\|y\|^{p-1}}{\|y\|^{p-1}} \geq$$

$$\geq \frac{|\,[x+y,y]\,|}{\|y\|^{p-1}} = \frac{\|x\|\,\|y\|^{p-1} + \|y\|^p}{\|y\|^{p-1}} = \|x\| + \|y\|,$$

i.e. $\|x+y\| = \|x\| + \|y\|$. Hence by strict convexity $y = \lambda x$ for some $\lambda > 0$.

Definition 5. *A s.i.p.s.(p) X is with the homogenity property if* $[x, \lambda y] = \lambda^{p-2}\overline{\lambda}[x,y]$ *for all* $x, y \in X$ *and all* $\lambda \in K$.

Theorem 3. *(Riesz representation). In a continuous s.i.p.s.(p) X with the homogenity property and which is uniformly convex and complete in the corresponding norm, for every functional* $f \in X'$ *there exists a unique element* $y \in X$ *such that*

$$f(x) = [x,y] \quad (x \in X).$$

Proof. For trivial functional $f(x) = 0$ $(x \in X)$ we take $y = 0$.

Suppose that $f(x) \neq 0$ for some $x \in X$ and denote $N = \{x : f(x) = 0\}$. Since $f \in X'$, N is a proper closed vector subspace of X. By Lemma 1. there exists an element $y_0 \neq 0$ normal to N.

We consider two cases:

a) For $x \in N$ we have $f(x) = [x,y] = 0$ for $y = \lambda x$ any scalar λ.

b) For $x = y_0$ taking

$$y = \frac{\overline{f(y_0)}}{|f(y_0)|^{\frac{p-2}{p-1}} \cdot \|y_0\|^{\frac{p}{p-1}}} \cdot y_0,$$

we obtain

$$[x,y] = [y_0, \frac{\overline{f(y_0)}}{|f(y_0)|^{\frac{p-2}{p-1}} \cdot \|y_0\|^{\frac{p}{p-1}}} \cdot y_0] =$$

$$= \frac{|\overline{f(y_0)}|^{p-2}}{|\overline{f(y_0)}|^{\frac{(p-2)^2}{p-1}} \|y_0\|^{\frac{p(p-2)}{p-1}}} \cdot \frac{f(y_0)}{|\overline{f(y_0)}|^{\frac{p-2}{p-1}} \|y_0\|^{\frac{p}{p-1}}} [y_0, y_0] =$$

$$= \frac{|\overline{f(y_0)}|^{p-2} \cdot f(y_0) \cdot \|y_0\|^p}{|f(y_0)|^{\frac{(p-2)^2+(p-2)}{p-1}} \cdot \|y\|^{\frac{p(p-2)+p}{p-1}}} = f(y_0).$$

Any $x \in X$ has the representation $x = z + \lambda y_0$ for $z \in N$ and y_0 normal to N and $\lambda = \frac{f(x)}{f(y_0)}$. Hence for all $x \in X$

$$f(x) = f(z) + \lambda f(y_0) = [z,y] + \lambda[y_0, y] = [x,y].$$

The obtained representation is unique. Namely, suppose that there exists at least two elements $y_1, y_2 \in X$ such that $y_1 \neq y_2$ and $f(x) = [x, y_1] = [x, y_2]$, for all $x \in X$. Taking $x = y_1$ we have $[y_1, y_1] = [y_1, y_2]$ which implies

$$\|y_1\|^p \leq \|y_1\| \cdot \|y_2\|^{p-1}, \ i.e. \ \|y_1\| \leq \|y_2\|.$$

On the orther side, taking $x = y_2$ we have $[y_2, y_1] = [y_2, y_2]$, which implies $\|y_2\|^p \leq \|y_2\| \cdot \|y_1\|^{p-1}$ *i.e.* $\|y_2\| \leq \|y_1\|$. Hence $\|y_1\| = \|y_2\|$.

This implies $\|y_1\|^p = \|y_1\| \cdot \|y_2\|^{p-1}$ and so

$$[y_1, y_2] = [y_1, y_1] = \|y_1\|^p = \|y_1\| \cdot \|y_2\|^{p-1}.$$

Since X is uniformly convex and so also strictly convex we have by Lemma 2. $y_1 = \lambda y_2$ for some $\lambda > 0$ and so $\|y_1\| = \lambda \|y_2\|$. Then $\|y_1\| = \|y_2\|$ implies $\lambda = 1$, *i.e.* $y_1 = y_2$.

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REZIME

O PROSTORIMA SA SEMI-SKALARNIM PROIZVODOM TIPA (p)

U radu se daje jednostavan dokaz činjenice da se svaki normiran vektorski prostor može svesti na prostor sa semi-skalarnim proizvodom tipa (p), koji je uveo B. Nath kao uopštenje Lumerovog prostora sa semi-skalarnim proizvodom. Uvodeći pojam homogenosti semi-skalarnog proizvoda tipa (p) dokazujemo Rieszovu teoremu o reprezentaciji linearne neprekidne funkcionale.

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