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SOME OPERATIONS ON THE SPACE $\mathcal{S}'^{(M_\alpha)}$ OF TEMPERED ULTRADISTRIBUTIONS

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Abstract

The space of test functions of the space $\mathcal{S}'^{(M_\alpha)}$ of tempered ultradisributions of Beurling type ([4]) is determined by several equivalent families of norms. Two representation theorems of the space $\mathcal{S}'^{(M_\alpha)}$ are proved. The operations of differentiation, ultradifferentiation and multiplication on $\mathcal{S}'^{(M_\alpha)}$ are investigated and the space of multipliers of the space $\mathcal{S}'^{(M_\alpha)}$ is determined.

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1. Introduction

In the paper we determine $\mathcal{S}^{(M_\alpha)}$, the space of test functions of the space $\mathcal{S}'^{(M_\alpha)}$ of tempered ultradisributions of Beurling type ([4]), by several equivalent families of norms, give two representation theorems of the space $\mathcal{S}'^{(M_\alpha)}$, investigate the operations of differentiation, ultradifferentiation and multiplication on $\mathcal{S}'^{(M_\alpha)}$ and determine the space of multipliers of the space $\mathcal{S}'^{(M_\alpha)}$. From the results of the paper and [4] it follows that $\mathcal{S}'^{(M_\alpha)}$ is a natural generalization of the space of Schwartz tempered distributions and of the space Σ'_s, $s > 1/2$ (see [6]). In fact, in the special case when (M_α) is Gevrey's sequence $(\alpha^{s\alpha})$ we have $\mathcal{S}'^{(M_\alpha)} = \Sigma'_s$.

Notation

The sets of non-negative integers, real and complex numbers are denoted by $\mathbf{N}$, $\mathbf{R}$ and $\mathbf{C}$. The usual norm on the space $L^p = L^p(\mathbf{R})$, $p \in [1,\infty]$, is denoted by $\|\cdot\|_p$.

We denote

$$\langle x\rangle^\beta = (1+x^2)^{\beta/2},\ \beta\in\mathbf{N}, x\in\mathbf{R} \ \text{ and } \ D = \frac{1}{i}\frac{\partial}{\partial x},\ i=\sqrt{-1}.$$

The letter C (without super- or subscript) will always denote a positive constant, not necessarily the same at each occurrence.

"$A \hookrightarrow B$" denotes that the inclusion mapping of the space A into the space B is continuous and that A is dense in B.

The sequence of Hermite functions (h_n) is given by

$$h_n(x) = \frac{(-1)^n}{\sqrt[4]{\pi}\sqrt{2^n n!}}\exp(x^2/2)\,(\exp(-x^2))^{(n)},\ n\in\mathbf{N},\ x\in\mathbf{R}.$$

The Fourier transform is defined by

$$\mathcal{F}\varphi(\xi) = \int_{\mathbf{R}} e^{-ix\xi}\varphi(x)dx,\ \xi\in\mathbf{R},\ \varphi\in L^1.$$

By (M_α) we denote a sequence of positive numbers which satisfies some of the following conditions (see [3])

(M.1) (*logarithmic convexity*)

$$M_\alpha^2 \le M_{\alpha-1}\,M_{\alpha+1},\ \ \alpha\in\mathbf{N}\backslash\{0\};$$

(M.2)' (*stability under differential operators*)

$$M_{\alpha+1} \le AH^\alpha M_\alpha,\ \ \alpha\in\mathbf{N},\ \text{ for some } A,H\ge 0;$$

(M.2) (*stability under ultradifferential operators*)

$$M_\alpha \le AH^\alpha \min_{0\le\beta\le\alpha} M_{\alpha-\beta}M_\beta,\ \ \alpha,\beta\in\mathbf{N},\ \text{for some } A,H\ge 0;$$

(M.3)' (*non-quasi-analyticity*)

$$\sum_{\alpha=1}^{\infty}\frac{M_{\alpha-1}}{M_\alpha} < \infty;$$

and

(M.3) (*strong non-quasi-analyticity*)

$$\sum_{\alpha=\beta+1}^{\infty} \frac{M_{\alpha-1}}{M_\alpha} \leq A\beta \frac{M_\beta}{M_{\beta+1}}, \quad \beta \in \mathbf{N}\backslash\{0\}.$$

We will always assume (M.1), (M.3)' and $M_0 = 1$. In some assertions we will suppose (M.2)', (M.2) and (M.3), as well. Throughout the paper the letters A and H will always denote the constants mentioned in (M.2)' and (M.2).

The so-called associated function for the sequence (M_α) is defined by

$$M(\rho) = \sup_\alpha \log \frac{\rho^\alpha}{M_\alpha}, \quad \rho > 0.$$

For the definitions and properties of the spaces $\mathcal{D}' = \mathcal{D}'(\mathbf{R})$, $\mathcal{S}' = \mathcal{S}'(\mathbf{R})$, $\mathcal{E}' = \mathcal{E}'(\mathbf{R})$, $\mathcal{D}'^{(M_\alpha)} = \mathcal{D}'^{(M_\alpha)}(\mathbf{R})$, and $\mathcal{E}'^{(M_\alpha)} = \mathcal{E}'^{(M_\alpha)}(\mathbf{R})$ we refer to [5] and [3].

2. Space $\mathcal{S}^{(M_\alpha)}$

Let $m > 0$ and $p \in [1, \infty)$ be given.

Definition 2.1. $\mathcal{S}_p^{M_\alpha, m}$ *and* $\mathcal{S}_\infty^{M_\alpha, m}$ *respectively are the spaces of all the smooth functions* φ *which satisfy*

$$\sigma_{m,p}(\varphi) = \left(\sum_{\alpha,\beta \in \mathbf{N}} \int_{\mathbf{R}} \mid \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} \langle x \rangle^\beta \varphi^{(\alpha)}(x) \mid^p dx \right)^{1/p} < \infty$$

and

$$\sigma_{m,\infty}(\varphi) = \sup_{\alpha,\beta \in \mathbf{N}} \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} \| \langle x \rangle^\beta \varphi^{(\alpha)} \|_\infty < \infty$$

respectively, equipped with the topology induced by the norms $\sigma_{m,p}$ *and* $\sigma_{m,\infty}$ *respectively.*

$$\mathcal{S}^{(M_\alpha)} = \lim \operatorname{proj}_{m\to\infty} \mathcal{S}_2^{M_\alpha, m}.$$

We will prove (see Theorem 2.2.) that if (M.2)' holds then

$$\mathcal{S}^{(M_\alpha)} = \lim \operatorname{proj}_{m\to\infty} \mathcal{S}_r^{M_\alpha,m}, \ r \in [1,\infty].$$

Note, the space $\mathcal{S}_r^{M_\alpha,m}$ is a special case of the space $\ell^r(m,F)$ ([8]). Using the analogous idea as in [8] one can prove that $\mathcal{S}_r^{M_\alpha,m}$ is a Banach space and especially, that $\mathcal{S}_2^{M_\alpha,m}$ is a Hilbert space where the scalar product is defined by

$$(\phi,\psi) = \sum_{\alpha,\beta\in\mathbf{N}} \int_{\mathbf{R}} \left(\frac{m^{\alpha+\beta}}{M_\alpha M_\beta}\right)^2 \langle x\rangle^{2\beta}\phi^{(\alpha)}(x)\overline{\psi^{(\alpha)}(x)}\,dx, \ \ \phi,\psi \in \mathcal{S}_2^{M_\alpha,m}.$$

The space $\mathcal{S}^{(M_\alpha)}$ is not trivial because under the assumptions (M.1) and (M.3)', the space $\mathcal{D}^{(M_\alpha)}$ is not trivial (see [3]) and $\mathcal{D}^{(M_\alpha)} \subset \mathcal{S}^{(M_\alpha)}$. Moreover, $\mathcal{S}^{(M_\alpha)}\setminus \mathcal{D}^{(M_\alpha)} \neq \emptyset$. If $\rho \in \mathcal{D}^{(M_\alpha)}$, $\rho \geq 0$, $supp\rho \subset [-1,1]$, $\rho(x) = 1$ for $x \in [-1/2,1/2]$ and (x_j) is a sequence of real numbers such that $|x_j| + 2 \leq |x_{j+1}|$, $j \in \mathbf{N}$, the function

$$\phi(x) = \sum_{j=1}^{\infty} \frac{\rho(x-x_j)}{\langle x_j\rangle^j}, \ \ x \in \mathbf{R}, \tag{1}$$

belongs to $\mathcal{S}^{(M_\alpha)}$ but it does not belong to $\mathcal{D}^{(M_\alpha)}$.

Since the inclusion mappings $i : \mathcal{S}^{M_\alpha,\tilde{m}} \longrightarrow \mathcal{S}^{M_\alpha,m}$, $0 < m \leq \tilde{m}$, are continuous it follows from above that $\mathcal{S}^{(M_\alpha)}$ is (FG)-space ([1]). Moreover, it is proved, in [4], that $\mathcal{S}^{(M_\alpha)}$ is an (F$\bar{\text{S}}$)-space ([1]), which implies that $\mathcal{S}^{(M_\alpha)}$ is a bornological, Fréchet, Montel and Schwartz space.

Theorem 2.2.

1. *The family of norms* $\{\sigma_{m,\infty}, m > 0\}$ *is equivalent to* $\{s_{m,\infty}, m > 0\}$, *where*

$$s_{m,\infty}(\varphi) = \sup_{\alpha,\beta\in\mathbf{N}} \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} \|x^\beta \varphi^{(\alpha)}\|_\infty.$$

2. *If (M.2)' holds and* $r,p \in [1,\infty]$, *the families of norms* $\{\sigma_{m,p},\ m > 0\}$ *and* $\{s_{m,p}, m > 0\}$ *are equivalent to* $\{\sigma_{m,r},\ m > 0\}$ *and* $\{s_{m,r}, m > 0\}$ *respectively, where*

$$s_{m,p}(\varphi) = \sum_{\alpha,\beta\in\mathbf{N}} \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} \|x^\beta \varphi^{(\alpha)}\|_p.$$

3. If (M.2)' holds the family of norms $\{s_{m,\infty}, m>0\}$ *is equivalent to* $\{\varsigma_m,\ m>0\}$, *where*

$$\varsigma_m(\varphi)=\sup_{\alpha\in\mathbf{N}}\frac{m^\alpha}{M_\alpha}\|\varphi^{(\alpha)}\exp(M(m\cdot))\|_\infty.$$

4. If (M.2) holds the family of norms $\{s_{m,2},\ m>0\}$ *is equivalent to any of the families of norms* $\{\theta_\delta, \delta>0\}$ *and* $\{\bar{s}_{m,2}, m>0\}$, *where*

$$\theta_\delta(\varphi)=\sum_{n\in\mathbf{N}}|a_n|^2\exp(2M(\delta\sqrt{2n+1})),\ \ a_n\in\mathbf{C},\ \ \varphi\overset{L^2}{=}\sum_{n\in\mathbf{N}}a_nh_n,$$

$$\bar{s}_{m,2}(\varphi)=\sum_{\alpha,\beta\in\mathbf{N}}\frac{m^{\alpha+\beta}}{M_\alpha M_\beta}\|(x^\beta\varphi)^{(\alpha)}\|_2.$$

Proof. Let us prove the first part of the theorem. Obviously for each smooth function φ and $m>0$, $s_{m,\infty}(\varphi)\le\sigma_{m,\infty}(\varphi)$. Since for each $L>0$

$$\frac{L^k k!}{M_k}\longrightarrow 0 \text{ as } k\to\infty, \tag{2}$$

which follows from (M.3)' (see [3, (4.5)]), and

$$\langle x\rangle^\beta\le 2^{\beta/2}max(1,|x|^\beta),\ x\in\mathbf{R},\ \beta\in\mathbf{N},$$

for each $m>0$ there exists C such that for each smooth function φ and $\alpha,\beta\in\mathbf{N}$

$$\frac{m^{\alpha+\beta}}{M_\alpha M_\beta}\|\langle x\rangle^\beta\varphi^{(\alpha)}\|_\infty\le\frac{m^{\alpha+\beta}}{M_\alpha M_\beta}2^\beta\, max(\|\varphi^{(\alpha)}\|_\infty,\|x^\beta\varphi^{(\alpha)}\|_\infty)\le$$

$$\le max(C\frac{m^\alpha}{M_\alpha}\|\varphi^{(\alpha)}\|_\infty,\frac{(2m)^{\alpha+\beta}}{M_\alpha M_\beta}\|x^\beta\varphi^{(\alpha)}\|_\infty)\le$$

$$\le C\sup_\beta\frac{(2m)^{\alpha+\beta}}{M_\alpha M_\beta}\|x^\beta\varphi^\alpha\|_\infty=Cs_{m,\infty}(\varphi).$$

Therefore for each $m>0$ there exists C such that for each smooth function φ, $\sigma_{m,\infty}(\varphi)\le Cs_{m,\infty}(\varphi)$.

Let us prove that $\{s_{m,p}, m > 0\}$ and $\{s_{m,r}, m > 0\}$ are equivalent families of norms. The proof of the equivalence of $\{\sigma_{m,p}, m > 0\}$ and $\{\sigma_{m,r}, m > 0\}$ is analogous. Let $t \in (1, \infty)$ and $\gamma = [1/t] + 1$. Applying (M.2)' we get that for each $m > 0$ there exists C such that for each smooth function φ

$$s_{m,t}(\varphi) \leq \sum_{\alpha,\beta} \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} \left(\sup_{|x|\leq 1} |x^\beta \varphi^{(\alpha)}(x)| + \right. \tag{3}$$

$$\left. + \sup_{|x|>1} |x^{\beta+\gamma}\varphi^{(\alpha)}| \int_{|x|>1} |x^{-\gamma}| dx \right) \leq$$

$$\leq \sum_{\alpha,\beta} \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} \|x^\beta \varphi^{(\alpha)}\|_\infty + C \sum_{\alpha,\beta} \frac{m^{\alpha+\beta} H^{\gamma\beta}}{M_\alpha M_{\beta+\gamma}} \|x^{\beta+\gamma}\varphi^{(\alpha)}\|_\infty \leq$$

$$\leq C s_{m(1+H^\gamma),\infty}(\varphi).$$

The inequality

$$|x^\beta \varphi^{(\alpha)}(x)| \leq \beta \int_R |t^\beta \varphi^{(\alpha)}(t)| dt + \int_R |t^\beta \varphi^{(\alpha+1)}(t)| dt, \quad x \in \mathbf{R} \quad \alpha, \beta \in \mathbf{N},$$

which holds for each smooth function φ, and condition (M.2)' imply that for each $m > 0$ there exists C such that for each smooth function φ

$$s_{m,\infty}(\varphi) \leq \sup_{\alpha,\beta} \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} (\beta \int_R |t^\beta \varphi^{(\alpha)}(t)| dt + \tag{4}$$

$$+ \int_R |t^\beta \varphi^{(\alpha+1)}(t)| dt) \leq$$

$$\leq C \sup_{\alpha,\beta} \left(\frac{2^\beta m^{\alpha+\beta}}{M_\alpha M_\beta} \int_R |t^\beta \varphi^{(\alpha)}(t)| dt + \frac{(Hm)^{\alpha+1} m^{\alpha+\beta+1}}{M_{\alpha+1} M_\beta} \int_R |t^\beta \varphi^{(\alpha+1)}(t)| dt \right) \leq$$

$$\leq C s_{2m(1+H),1}(\varphi).$$

Let $t \in (1,\infty)$, $q = t/(t-1)$ and $\gamma = [1/q]+1$. The Hölder inequality, (2) and (M.2)' imply that for each $m > 0$ there exists C such that for each smooth function φ

$$s_{m,1}(\varphi) = \sum_{\alpha,\beta} \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} \left(\int_{|x|\le 1} |\varphi^{(\alpha)}(x)|dx + \right. \tag{5}$$

$$\left. + \int_{|x|>1} |x^\beta \varphi^{(\alpha)}(x)|dx \right) \le$$

$$\le \sum_{\alpha,\beta} \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} \left(C \left(\int_{|x|\le 1} |\varphi^{(\alpha)}(x)|^t dx \right)^{1/t} + \right.$$

$$\left. + \left(\int_{|x|>1} |x^{\beta+\gamma}\varphi^{(\alpha)}(x)|^t dx \right)^{1/t} \left(\int_{|x|>1} |x|^{-\gamma q} dx \right)^{1/q} \right) \le$$

$$\le C \sum_{\alpha,\beta} \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} \left(\|\varphi^{(\alpha)}\|_t + \|x^{\beta+\gamma}\varphi^{(\alpha)}\|_t \right) \le$$

$$\le C \left(\sum_{\alpha} \frac{m^\alpha}{M_\alpha} \|\varphi^{(\alpha)}\|_t + \sum_{\alpha,\beta} \frac{m^{\alpha+\beta} H^{\gamma\beta}}{M_\alpha M_{\beta+\gamma}} \|x^{\beta+\gamma}\varphi^{(\alpha)}\|_t \right) \le$$

$$\le C s_{m(1+H^\gamma),t}(\varphi).$$

The equivalence of $\{\sigma_{m,r},\ m>0\}$ and $\{\sigma_{m,p},\ m>0\}$ follows from (3), (4) and (5).

Let us now prove the third part of the theorem. The condition (M.2)' implies that for each $\varphi \in \mathcal{S}^{(M_\alpha)}$ and $m > 0$ there exists C such that for each $\alpha, \beta \in \mathbf{N}$ and for $|x| > k > 1$

$$\frac{m^{\alpha+\beta}}{M_\alpha M_\beta} |x^\beta \varphi^{(\alpha)}(x)| \le C \frac{m^\alpha (mH)^{\beta+1}}{M_\alpha M_{\beta+1}} |x^\beta \varphi^{(\alpha)}(x)| \le$$

$$\leq \frac{C}{k} \frac{m^\alpha (mH)^{\beta+1}}{M_\alpha M_{\beta+1}} |x^{\beta+1} \varphi^{(\alpha)}(x)| \leq \frac{C}{k}.$$

Therefore for fixed $\varphi \in \mathcal{S}^{(M_\alpha)}$ and $m > 0$, $\frac{m^{\alpha+\beta}}{M_\alpha M_\beta} |x^\beta \varphi^{(\alpha)}(x)|$ converges uniformly in $\alpha, \beta \in \mathbf{N}$ to zero as $|x|$ tend to infinity. The definition of the space $\mathcal{S}^{(M_\alpha)}$ implies that if m and $(\alpha+\beta)$ tends to infinity, then $\frac{m^{\alpha+\beta}}{M_\alpha M_\beta} |x^\beta \varphi^{(\alpha)}(x)|$ converges to zero uniformly in $x \in \mathbf{R}$. Hence, for given $\varphi \in \mathcal{S}^{(M_\alpha)}$ and $m > 0$ there are $\alpha_0, \beta_0 \in \mathbf{N}$ and $x_0 \in \mathbf{R}$ such that

$$\sup_{\alpha,\beta} \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} \|x^\beta \varphi^{(\alpha)}\|_\infty = \frac{m^{\alpha_0+\beta_0}}{M_{\beta_0} M_{\alpha_0}} |x_0^{\beta_0} \varphi^{(\alpha_0)}(x_0)| =$$

$$= \sup_{\alpha,\beta} \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} \|x^\beta \varphi^{(\alpha)} \|_\infty = \| \sup_\beta (\sup_\alpha \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} |x^\beta \varphi^{(\alpha)}|) \|_\infty =$$

$$= \| \sup_\alpha (\sup_\beta \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} |x^\beta \varphi^{(\alpha)}|) \|_\infty = \sup_\alpha (\| \sup_\beta \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} |x^\beta \varphi^{(\alpha)}| \|_\infty) =$$

$$= \sup_{\alpha \in \mathbf{N}} (\frac{m^\alpha}{M_\alpha} \|\varphi^{(\alpha)} \exp(M(m\cdot))\|_\infty).$$

The proof of the fourth part of the theorem is given in [4]. □

Theorem 2.3. *If (M.2)' is fulfilled then*

$$\mathcal{D}^{(M_\alpha)} \hookrightarrow \mathcal{S}^{(M_\alpha)} \hookrightarrow \mathcal{E}^{(M_\alpha)} \quad \text{and} \quad \mathcal{S}^{(M_\alpha)} \hookrightarrow \mathcal{S}.$$

Proof. Let $\varphi \in \mathcal{D}^{(M_\alpha)}$ and $supp\varphi \subset [-k, k]$, $k > 1$. The condition (M.3)' implies that for each $m > 0$ there exists C such that

$$\sup_{\alpha,\beta} \frac{m^{\alpha+\beta}}{M_\alpha M_\alpha} \|\langle x\rangle^\beta \varphi^{(\alpha)}\|_\infty = \sup_{\alpha,\beta} \frac{(mk)^\beta m^\alpha}{M_\beta M_\alpha} \|\varphi^{(\alpha)}\|_\infty \leq C \sup_\alpha \frac{m^\alpha}{M_\alpha} \|\varphi^{(\alpha)}\|_\infty.$$

It follows that the inclusion mapping $i : \mathcal{D}^{(M_\alpha)} \longrightarrow \mathcal{S}^{(M_\alpha)}$ is continuous.

Since for fixed $\varphi \in \mathcal{S}^{(M_\alpha)}$ and $m > 0$, $\frac{m^{\alpha+\beta}}{M_\alpha M_\beta}|x^\beta \varphi^{(\alpha)}(x)|$ converges uniformly in $\alpha, \beta \in \mathbf{N}$ as $|x|$ tends to infinity (see the proof of Theorem 2.2.), the sequence $(\varphi_j)_j$, where $\varphi_j(x) = \rho(x/j)\, \rho(x)$ and ρ is a function defined by (1) converges to φ in the space $\mathcal{S}^{(M_\alpha)}$. It follows that $\mathcal{D}^{(M_\alpha)}$ is dense in $\mathcal{S}^{(M_\alpha)}$. □

Let

$$P^*(x, D) = \sum_{\mu,\nu \in \mathbf{N}} a_{\mu,\nu}\, (-1)^\nu D^\nu x^\mu,$$

where $a_{\mu,\nu}$ are complex numbers which satisfy that there exist $L > 0$ and C such that

$$|a_{\mu,\nu}| \le C \frac{L^{\mu+\nu}}{M_\mu M_\nu}, \quad \mu, \nu \in \mathbf{N}, \tag{6}$$

Theorem 2.4.

1. *If (M.2)' is fulfilled*

$$(-1)^\nu D^\nu : \mathcal{S}^{(M_\alpha)} \longrightarrow \mathcal{S}^{(M_\alpha)}, \quad \varphi \mapsto (-1)^\nu D^\nu \varphi, \quad \nu \in \mathbf{N}, \tag{7}$$

and

$$P^*(x, D) : \mathcal{S}^{(M_\alpha)} \longrightarrow \mathcal{S}, \quad \varphi \mapsto P^*(x, D)\varphi, \tag{8}$$

are continuous linear mappings.

2. *If (M.2) is fulfilled*

$$P^*(x, D) : \mathcal{S}^{(M_\alpha)} \longrightarrow \mathcal{S}^{(M_\alpha)}, \quad \varphi \mapsto P^*(x, D)\varphi, \tag{9}$$

is a continuous linear mapping.

3. *The family of translation operators*

$$\tau_h : \mathcal{S}^{(M_\alpha)} \longrightarrow \mathcal{S}^{(M_\alpha)}, \quad \tau_h : \varphi(\cdot) \mapsto \varphi(\cdot - h), \quad |h| \le h_0,$$

where $h_0 > 0$, is uniformly continuous.

Proof. Let us prove that (8) is a continuous mapping. Applying (6), (M.2)', (M.1), and (M.3)', we get that for $\varphi \in \mathcal{S}^{(M_\alpha)}$ and $\alpha, \beta \in \mathbf{N}$

$$\|x^\beta (P^*(x,D)\varphi)^{(\alpha)}\|_\infty \leq$$

$$\leq C \sum_{\mu,\nu \in \mathbf{N}} \sum_{k=0}^{\min(\alpha+\nu,\beta)} \binom{\alpha+\nu}{k} \frac{L^{\nu+\mu}}{M_\nu M_\mu}, \|((x^\beta)^{(k)} x^\mu \varphi)^{(\alpha+\nu-k)}\|_\infty \leq$$

$$\leq C \sum_{\mu,\nu \in \mathbf{N}} (\sum_{k=0}^{\min(\alpha+\nu,\beta)} \binom{\alpha+\nu}{k} \binom{\beta}{k} k! \frac{H^{\alpha\nu} H^{\beta\mu} L^{\nu+\mu}}{M_{\nu+\alpha} M_{\mu+\beta}} \|(x^{\mu+\beta-k}\varphi)^{(\nu+\alpha-k)}\|_\infty) \leq$$

$$\leq C \sum_{\mu,\nu \in \mathbf{N}} \sum_{k=0}^{\min(\alpha+\nu,\beta)} \binom{\alpha+\nu}{k} \binom{\beta}{k} \frac{1}{4^{\mu+\nu}} \frac{((1+4L)(1+H^\alpha)(1+H^\beta))^{2k} k!}{M_k} \cdot$$

$$\cdot \frac{((1+4L)(1+H^\alpha)(1+H^\beta))^{\mu+\nu+\alpha+\beta-2k}}{M_{\nu+\alpha-k} \, M_{\mu+\beta-k}} \|(x^{\mu+\beta-k} \, \varphi)^{(\nu+\alpha-k)}\|_\infty \leq$$

$$\leq C \sup_{\vartheta,\eta} \frac{((1+4L)(1+H^\alpha)(1+H^\beta))^{\vartheta+\eta}}{M_\vartheta M_\eta} \|(x^\eta \varphi)^{(\vartheta)}\|_\infty.$$

This implies the continuity of (8).

Let us prove that (9) is a continuous mapping. Applying respectively (M.2), (M.1) and (M.3)' we get that for each $m > 0$ there exists C such that for each $\varphi \in \mathcal{S}^{(M_\alpha)}$

$$\sup_{\alpha,\beta} \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} \|x^\beta (P^*(x,D)\varphi)^{(\alpha)}\|_\infty \leq$$

$$\leq C \sup_{\alpha,\beta} \sum_{\mu,\nu \in \mathbf{N}} \sum_{k=0}^{\min(\alpha+\nu,\beta)} \binom{\alpha+\nu}{k} \binom{\beta}{k} k! \frac{H^{\nu+\alpha} H^{\mu+\beta} m^{\beta+\alpha}}{M_{\nu+\alpha} M_{\mu+\beta}} L^{\mu+\nu}.$$

$$\cdot \|(x^{\mu+\beta-k}\varphi)^{(\nu+\alpha-k)}\|_\infty \leq$$

$$\leq \mathcal{C} \sup_{\alpha,\beta} \sum_{\mu,\nu\in\mathbf{N}} \sum_{k=0}^{\min(\alpha+\nu,\beta)} \frac{1}{8^{\alpha+\beta+\mu+\nu}} \binom{\alpha+\nu}{k}\binom{\beta}{k} \cdot$$

$$\cdot \frac{k!(8mL(1+H))^{2k}}{M_k M_k} \frac{(8mL(1+H))^{\alpha+\beta+\mu+\nu-2k}}{M_{\nu+\alpha-k}M_{\mu+\beta-k}} \|(x^{\mu+\beta-k}\varphi)^{(\nu+\alpha-k)}\|_\infty \leq$$

$$\leq \mathcal{C} \sup_{\alpha,\beta} \sum_{\mu,\nu\in\mathbf{N}} \sum_{k=0}^{\min(\alpha+\nu,\beta)} \frac{1}{8^{\alpha+\beta+\mu+\nu}} \frac{(16mL(1+H))^{\alpha+\beta+\mu+\nu-2k}}{M_{\nu+\alpha-k}M_{\mu+\beta-k}} \|(x^{\mu+\beta-k}\varphi)^{(\nu+\alpha-k)}\|_\infty \leq$$

$$\leq \mathcal{C} \sup_{\alpha,\beta} \frac{(16mL(1+H))^{\beta+\alpha}}{M_\beta M_\alpha} \|(x^\beta\varphi)^{(\alpha)}\|_\infty.$$

This implies the continuity of (9).

Let us now prove the fourth part of the theorem. If $m > 0$, $\varphi \in \mathcal{S}^{(M_\alpha)}$ and $|h| \leq h_0$, we have

$$\sup_{\alpha,\beta} \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} \|\langle x\rangle^\beta (\tau_h\varphi)^{(\alpha)}\|_\infty \leq$$

$$\leq \sup_{\alpha,\beta} \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} \sup_{x\in\mathbf{R}} |\langle x-h\rangle^\beta \varphi^{(\alpha)}(x)| \leq \sup_{\alpha,\beta} \frac{(2\langle h_0\rangle m)^{\alpha+\beta}}{M_\alpha M_\beta} \|\langle x\rangle^\beta \varphi^{(\alpha)}\|_\infty. \square$$

3. The space of tempered ultradistributions $\mathcal{S}'^{(M_\alpha)}$

Definition 3.1. *The space $\mathcal{S}'^{(M_\alpha)}$ of tempered ultradistributions of Beurling type is the strong dual of $\mathcal{S}^{(M_\alpha)}$.*

A non-trivial example of an element of the space $\mathcal{S}'^{(M_\alpha)}$ is defined by

$$\langle f, \varphi\rangle = \int_{\mathbf{R}} f\varphi dx, \quad \varphi \in \mathcal{S}^{(M_\alpha)},$$

where f is an ultradifferentiable function of class (M_α) (see [3]) and that there exist $L > 0$ and C such that

$$|f(x)| \leq C \sum_\beta \frac{L^\beta}{M_\beta} x^\beta, \quad x \in \mathbf{R}.$$

$\mathcal{S}'^{(M_\alpha)}$ is a separable, complete and Montel space (see [4]).

Note,

$$\mathcal{S}' \hookrightarrow \mathcal{S}'^{(M_\alpha)} \text{ and } \mathcal{E}'^{(M_\alpha)} \hookrightarrow \mathcal{S}'^{(M_\alpha)} \hookrightarrow \mathcal{D}'^{(M_\alpha)}.$$

follows from Theorem 2.3..

Theorem 3.2. *Let* $f \in \mathcal{S}'^{(M_\alpha)}$ *and* $r \in (1, \infty]$.

1. *There exists a sequence of functions* $(F_{\alpha,\beta})_{\alpha,\beta\in\mathbf{N}}$ *from* L^r *such that in the space* $\mathcal{S}'^{(M_\alpha)}$

$$f = \sum_{\alpha,\beta} (\langle x\rangle^\beta F_{\alpha,\beta})^{(\alpha)}, \tag{10}$$

and that for some $m > 0$

$$\begin{cases} \left(\sum_{\alpha,\beta} \int_{\mathbf{R}} |\frac{M_\alpha M_\beta}{m^{\alpha+\beta}} F_{\alpha,\beta}(x)|^r\right)^{1/r} < \infty, & r \in (1,\infty), \\ \sup_{\substack{\alpha,\beta \\ x\in\mathbf{R}}} \left(\frac{M_\alpha M_\beta}{m^{\alpha+\beta}} |F_{\alpha,\beta}(x)|\right) < \infty, & r = \infty. \end{cases} \tag{11}$$

2. *If for a sequence* $(F_{\alpha,\beta})_{\alpha,\beta\in\mathbf{N}}$ *from* L^r *holds (11), then the sum on the right hand side of (10) converges in* $\mathcal{S}'^{(M_\alpha)}$.

3. If $f \in \mathcal{S}'^{(M_\alpha)}$ and $a_n = \langle f, h_n \rangle$, $n \in \mathbf{N}$, then

$$f = \sum_{n\in\mathbf{N}} a_n h_n,$$

in $\mathcal{S}'^{(M_\alpha)}$ and for some $\delta > 0$

$$\sum_{n\in\mathbf{N}} |a_n|^2 \exp(-2M(\delta\sqrt{2n+1})) < \infty. \tag{12}$$

4. Let $(a_n)_n$ be a sequence of complex numbers. The series $\sum_{n\in\mathbf{N}} a_n h_n$ converges in $\mathcal{S}'^{(M_\alpha)}$ if and only if (12) holds for some $\delta > 0$. If f is the sum of the series then $a_n = \langle f, h_n \rangle$, $n \in \mathbf{N}$.

Note, the weak and the strong sequential convergence are equivalent in $\mathcal{S}'^{(M_\alpha)}$.

Proof. The proof of the theorem is analogous to the proof of [6, Theorem 5.2.]. Let $p = r/(r-1)$. Note, $p \in [1, \infty)$. Since $\mathcal{S}'^{(M_\alpha)}$ is a strict (F$\bar{\text{S}}$)-space, we have

$$\mathcal{S}'^{(M_\alpha)} = \text{ind} \lim_{m\to\infty} \left(\overline{\mathcal{S}_p^{M_\alpha,m}}\right)',$$

in the sense of strong topologies, where $\overline{\mathcal{S}_p^{M_\alpha,m}}$ is the closure of $\mathcal{S}^{(M_\alpha)}$ in the space $\mathcal{S}_p^{M_\alpha,m}$, with the topology induced by the space $\mathcal{S}_p^{M_\alpha,m}$.

If $f \in \mathcal{S}'^{(M_\alpha)}$ there exists $m > 0$ such that f has a continuous, linear extension on $\overline{\mathcal{S}_p^{M_\alpha,m}}$. The Hahn-Banach theorem implies that f has a continuous, linear extension on $\overline{\mathcal{S}_p^{M_\alpha,m}}$ with the same dual norm. We denote this extension again by f. Let $T_p(m)$ be the space of sequences $(\psi_{\alpha,\beta})_{\alpha,\beta\in\mathbf{N}}$ from $L^r(\mathbf{R})$ equipped with the norm

$$\|(\psi_{\alpha,\beta})_{\alpha,\beta}\| = \left(\sum_{\alpha,\beta} \int_{\mathbf{R}} |\frac{m^{\alpha+\beta}}{M_\beta M_\alpha} \psi_{\alpha,\beta}|^p dx\right)^{1/p} < \infty.$$

The mapping

$$i : \overline{\mathcal{S}_p^{M_\alpha m}} \to T_p(m) \quad i : \varphi \mapsto ((-1)^\alpha \langle x \rangle^\beta \varphi^{(\alpha)})_{\alpha,\beta}$$

is an isometry of $\overline{\mathcal{S}_p^{M_\alpha m}}$ onto $G_p(m) = i\left(\overline{\mathcal{S}_p^{M_\alpha m}}\right) \subset T_p(m)$. We define a continuous linear functional $\tilde{f}$ on $G_p(m)$ by

$$\langle \tilde{f}, (\psi_{\alpha,\beta})_{\alpha,\beta} \rangle = \langle f, i^{-1}((\psi_{\alpha,\beta})_{\alpha,\beta}) \rangle, \ (\psi_{\alpha,\beta})_{\alpha,\beta} \in G_p(m).$$

Again by the Hahn-Banach theorem we extended $\tilde{f}$ linearly and continuously on $T_p(m)$ with the same dual norm, and denote this extension by F.

It is known (see [8, p.29, Hilfsatz 2.]) that the fact $F \in (T_p(m))'$ implies the existence of a sequence $(F_{\alpha,\beta})_{\alpha,\beta\in\mathbf{N}}$ from L^r such that F has a form

$$\langle F, (\psi_{\alpha,\beta})_{\alpha,\beta}\rangle = \sum_{\alpha,\beta}\int_{\mathbf{R}} F_{\alpha,\beta}(x)\psi_{\alpha,\beta}(x)dx,\ ((\psi_{\alpha,\beta})_{\alpha,\beta}) \in T_p(m)$$

and that the norm of F is given by

$$\|F\| = \begin{cases} (\sum_{\alpha,\beta}\int_{\mathbf{R}} |\frac{M_\alpha M_\beta}{m^{\alpha+\beta}} F_{\alpha,\beta}(x)|^r)^{1/r} < \infty \text{ if } r \in (1,\infty), \\ \sup_{\substack{\alpha,\beta \\ x\in\mathbf{R}}} \frac{M_\alpha M_\beta}{m^{\alpha+\beta}}|F_{\alpha,\beta}(x)| < \infty \text{ if } r = \infty. \end{cases}$$

Thus $\|F\| = \|f\| < \infty$ and for each $\varphi \in \mathcal{S}^{(M_\alpha)}$ we have

$$\langle f,\varphi\rangle = \langle \tilde{f}, ((-1)^\alpha \langle x\rangle^\beta \varphi^{(\alpha)}))_{\alpha,\beta}\rangle \ = \ \langle F, ((-1)^\alpha \langle x\rangle^\beta \varphi^{(\alpha)})_{\alpha,\beta}\rangle =$$

$$= \sum_{\alpha,\beta}(-1)^\alpha \int_{\mathbf{R}} F_{\alpha,\beta}(x)\langle x\rangle^\beta \varphi^{(\alpha)}(x)dx \ = \ \sum_{\alpha,\beta}\langle ((\langle x\rangle^\alpha\ F_{\alpha,\beta})^{(\beta)}, \varphi\rangle,$$

which implies the first part of the theorem.

Let us now prove the third part of the theorem. For each $\varphi \overset{L^2}{=} \sum_n b_n h_n$ an element of $\mathcal{S}^{(M_\alpha)}$ we have

$$\langle \sum_n a_n h_n, \varphi\rangle = \sum_n a_n \langle h_n,\varphi\rangle = \sum_n \langle f, h_n\rangle b_n = \langle f, \sum_n b_n h_n\rangle = \langle f,\varphi\rangle.$$

It is easy to check, applying Theorem 2.2. part 4, that $\sum_n \exp(-M(\delta\sqrt{2n+1}))h_n$ is an element of $\mathcal{S}^{(M_\alpha)}$. It follows

$$\sum_n |a_n|^2 \exp(-2M(\delta\sqrt{2n+1})) =$$

$$= \sum_n |\langle f, \exp(-M(\delta\sqrt{2n+1}))h_n\rangle|^2 \leq |\langle f, \sum_n \exp(-M(\delta\sqrt{2n+1}))h_n\rangle|^2 < \infty.$$

Let $\varphi \overset{L^2}{=} \sum_n b_n h_n$ be an element of the space $\mathcal{S}^{(M_\alpha)}$ and let $\delta > 0$ be such that $\sum_n |b_n|^2 \exp(2M(\delta\sqrt{2n+1})) < \infty$ (see Theorem 2.2.). The fourth part of the theorem follows from the estimations

$$|\langle \sum_n h_n a_n, \varphi\rangle|^2 = |\sum_n a_n b_n|^2 \leq$$

$$\leq \sum_n |a_n|^2 \exp(-2M(\delta\sqrt{2n+1})) \sum_n |b_n|^2 \exp(2M(\delta\sqrt{2n+1})) < \infty. \square$$

An immediate consequence of the above theorem is that the linear hull of $\{h_n,\ n \in \mathbf{N}\}$ is a dense subspace of $\mathcal{S}'^{(M_\alpha)}$.

Theorem 3.3.

1. *Suppose (M.2)'. The operators*

$$D^\mu : \mathcal{S}'^{(M_\alpha)} \longrightarrow \mathcal{S}'^{(M_\alpha)}, \quad \nu \in \mathbf{N}, \tag{13}$$

$$P(x, D) : \mathcal{S}' \longrightarrow \mathcal{S}'^{(M_\alpha)}, \tag{14}$$

   defined as the adjoints of (7), (8) respectively, are continuous and for each $f \in \mathcal{S}'^{(M_\alpha)}$ we have

$$P(x, D)f = \sum_{\mu,\nu\in\mathbf{N}} a_{\mu,\nu}\, x^\mu D^\nu f, \tag{15}$$

   where the series on the right hand side converge absolutely in $\mathcal{S}'^{(M_\alpha)}$.

2. *Suppose (M.2). The operator*

$$P(x, D) : \mathcal{S}'^{(M_\alpha)} \longrightarrow \mathcal{S}'^{(M_\alpha)},$$

   defined as the adjoint of (9) is continuous and for each $f \in \mathcal{S}'^{(M_\alpha)}$, we have (15).

Proof. The continuity of all the mentioned mappings follows from Theorem 2.4..

Suppose (M.2)' resp. (M.2). Let $f \in \mathcal{S}'$ resp. $f \in \mathcal{S}'^{(M_\alpha)}$. Since for each $\varphi \in \mathcal{S}^{(M_\alpha)}$

$$\langle f, \sum_{\mu,\nu \le n} a_{\mu,\nu}(-1)^\nu D^\nu x^\mu \varepsilon\rangle = \langle \sum_{\mu,\nu\le n} a_{\mu,\nu} x^\mu D^\nu f, \varphi\rangle$$

converges to

$$\langle f, P^*(x,D)\varphi\rangle = \langle \sum_{\mu,\nu\le n} a_{\mu,\nu} x^\mu D^\nu f, \varphi\rangle,$$

as $n \to \infty$, we have (15). □

4. The space $\mathcal{O}_M^{(M_\alpha)}$ of multipliers of $\mathcal{S}'^{(M_\alpha)}$

Definition 4.1. *$\mathcal{O}_M^{(M_\alpha)}$ is the space of all $\varphi \in \mathcal{E}^{(M_\alpha)}$ such that for all $\psi \in \mathcal{S}^{(M_\alpha)}$ the pointwiese product $\varphi \cdot \psi$ belongs to $\mathcal{S}^{(M_\alpha)}$, wich topology is induced by the family of seminorms*

$$p_{\psi,m}(\varphi) = \sup_{\alpha,\beta} \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} \|\langle x\rangle^\beta (\psi\varphi)^{(\alpha)}\|_\infty, \quad \psi \in \mathcal{S}^{(M_\alpha)}, \ m > 0.$$

The inclusion mappings

$$\mathcal{S}^{(M_\alpha)} \longrightarrow \mathcal{O}_M^{(M_\alpha)} \longrightarrow \mathcal{S}'^{(M_\alpha)}$$

are continuous. Moreover, $\mathcal{S}^{(M_\alpha)}$ is dense in $\mathcal{O}_M^{(M_\alpha)}$.

Theorem 4.2. *Let $\varphi \in \mathcal{E}^{(M_\alpha)}$.*

1. The condition

(a) for all $\psi \in \mathcal{S}^{(M_\alpha)}$, the pointwiese product $\varphi\psi \in \mathcal{S}^{(M_\alpha)}$;

implies

(b) for every $m>0$ there exist C and $\ell>0$, such that

$$\sup_\alpha \frac{m^\alpha}{M_\alpha}|\varphi^{(\alpha)}(x)| \le C\sum_\beta \frac{\ell^\beta}{M_\beta}\langle x\rangle^\beta,\ x\in\mathbf{R}; \tag{16}$$

2. If (M.2) is fulfilled all above conditions are equivalent.

Proof. If (1a) is and (1b) is not fulfilled then for some $m>0$ there exists a sequence $(x_j)_j$ such that $|x_j|$ tends to infinity as $j\to\infty$ and

$$\sup_\alpha \frac{m^\alpha}{M_\alpha}\,|\varphi^{(\alpha)}(x_j)| > M_j\sum_\beta \frac{1}{M_\beta}\langle x_j\rangle^\beta. \tag{17}$$

Without the loss of generality we may suppose that $|x_j|+2\le |x_j+1|$, $j\in\mathbf{N}$. Consider the function $\phi\in\mathcal{S}^{(M_\alpha)}$, defined by (1). The conditions (M.1) and (M.3)' imply

$$\sup_\alpha \frac{m^\alpha}{M_\alpha}|(\phi\varphi)^{(\alpha)}(x_j)| = \sup_\alpha \frac{m^\alpha}{M_\alpha}\sum_{k=0}^{\alpha}\binom{\alpha}{k}|\frac{\rho^{(k)}(0)}{\langle x_j\rangle^j}\,\varphi^{(\alpha-k)}(x_j)| =$$

$$= \sup_\alpha \frac{m^\alpha}{M_\alpha}|\,\frac{\rho(0)}{\langle x_j\rangle^j}\varphi^{(\alpha)}(x_j)| > M_j\sum_{\beta\ge j}\frac{1}{M_\beta}\,\langle x_j\rangle^{\beta-j} \ge$$

$$\ge M_j\sum_{\beta\ge j}\frac{1}{M_{\beta-j}M_j}\,\langle x_j\rangle^{\beta-j} = \sum_\beta \frac{m^\beta}{M_\beta}\langle x_j\rangle^\beta > 1.$$

Hence, $\sup_\alpha \frac{m^\alpha}{M_\alpha}|(\phi\varphi)^\alpha(x_j)|$ does not converge to zero as $|x_j|\to\infty$, which is a contradiction (see the proof of Theorem 2.2.).

Let (1b) and (M.2) be fulfilled and let $\psi\in\mathcal{S}^{(M_\alpha)}$. We will prove that $\varphi\psi\in\mathcal{S}^{(M_\alpha)}$. The conditions (1b) and (M.2) imply that for each $m>0$ there exist C and $\ell>1$ such that

$$\sup_{\alpha,\beta}\frac{m^{\alpha+\beta}}{M_\alpha M_\beta}\|\langle x\rangle^\beta(\psi\varphi)^{(\alpha)}\|_\infty \le \sup_{\alpha,\beta}\sum_{k\le\alpha}\binom{\alpha}{k}\frac{m^{\alpha+\beta}}{M_\alpha M_\beta}\,\|\langle x\rangle^\beta\psi^{(k)}\varphi^{(\alpha-k)}\|_\infty \le$$

$$\leq C \sup_{\alpha,\beta} \sum_{\gamma} \frac{(2m)^{\alpha+\beta} \ell^{\gamma}}{M_\alpha M_\beta M_\gamma} \|\langle x \rangle^{\beta+\gamma} \varphi^{(\alpha)}\|_\infty \leq$$

$$\leq C \sup_{\alpha,\beta} \sum_{\gamma} \frac{1}{2^\gamma} \frac{(2m)^{\alpha+\beta} (2\ell)^{\gamma} H^{\beta+\gamma}}{M_\alpha M_{\beta+\gamma}} \|\langle x \rangle^{\beta+\gamma} \varphi^{(\alpha)}\|_\infty \leq$$

$$\leq C \sup_{\alpha,\beta} \frac{(4m\ell(1+H))^{\alpha+\beta}}{M_\alpha M_\beta} \|\langle x \rangle^{\beta} \varphi^{(\alpha)}\|_\infty . \square$$

The next theorem follows from the proof of Theorem 4.2..

Theorem 4.3.

1. *The mappings*

$$\mathcal{O}_M^{(M_\alpha)} \longrightarrow \mathcal{S}^{(M_\alpha)}, \ \ \varphi \mapsto \psi\varphi, \ \ \psi \in \mathcal{S}^{(M_\alpha)},$$

are continuous.

2. *Suppose (M.2). The pointwiese multiplication*

$$\mathcal{S}^{(M_\alpha)} \times \mathcal{O}_M^{(M_\alpha)} \longrightarrow \mathcal{S}^{(M_\alpha)}, \ \ (\psi, \varphi) \mapsto \psi\varphi,$$

is a separately continuous mapping.

3. *$\mathcal{S}^{(M_\alpha)}$ in 1. and 2. may be replaced by $\mathcal{S}'^{(M_\alpha)}$.*

Theorem 4.4. *If $\phi \in \mathcal{E}^{(M_\alpha)}$ and for all $f \in \mathcal{S}'^{(M_\alpha)}$ the product ϕf belongs to $\mathcal{S}'^{(M_\alpha)}$, then ϕ belongs to $\mathcal{O}_M^{(M_\alpha)}$.*

Proof. Our assumption implies that for every $\varphi \in \mathcal{S}^{(M_\alpha)}$ the mapping

$$f \mapsto \langle \phi f, \varphi \rangle$$

is continuous linear functional on $\mathcal{S}'^{(M_\alpha)}$. Since $\mathcal{S}'^{(M_\alpha)}$ is a reflexive space, there is $\psi \in \mathcal{S}^{(M_\alpha)}$ such that for each $f \in \mathcal{S}'^{(M_\alpha)}$,

$$\langle \phi f, \varphi \rangle = \langle f, \psi \rangle.$$

In particular, for each $\rho \in \mathcal{D}^{(M_\alpha)}$, we have

$$\langle \phi \rho, \varphi \rangle = \langle \rho, \psi \rangle,$$

which implies that

$$\langle \rho, \phi \varphi \rangle = \langle \rho, \psi \rangle.$$

Hence for all $\varphi \in \mathcal{S}^{(M_\alpha)}$ we have $\phi\varphi = \psi \in \mathcal{S}^{(M_\alpha)}$. It follows $\phi \in \mathcal{O}_M^{(M_\alpha)}$. □

References

[1] Floret K., Wloka J., Einführung in die Theorie der lokalkonvexen Räume, Lecture Notes in Mathematics 56, Berlin-Heilderberg-New York, Springer Verlag, 1968.

[2] Gel'fand I.M., Shilov G.E., Generalized Functions, Volume 2, New York and London, Academic Press, 1968.

[3] Komatsu H., ULtradistributions I, J. Fac. Sci. Univ. Tokyo Sect. IA Mat. 20 (25-105) (1973).

[4] Kovačević D., Pilipović S., Integral Transformations on the Space of Tempered Ultradistributions, preprint (1991).

[5] Schwartz L., Théorie des distributions, Paris, Hermann, 1966.

[6] Pilipović S., Tempered Ultradistributions, Bollettiono U.M.I. (7) 2-B, 235-251 (1988).

[7] Pilipović S., Some Operations in Σ'_s, $s > 1/2$, Radovi Matematički, Vol. 5, 53-62 (1989).

[8] Wloka J., Grundraume und veralgemenete Funktionen, Lecture Notes in Mathematics 82, Berlin-Heilderberg-New York, Springer Verlag, (1969).

REZIME

NEKE OPERACIJE U PROSTORU $\mathcal{S}'^{(M_\alpha)}$ TEMPERIRANIH ULTRADISTRIBUCIJA

Prostor $\mathcal{S}^{(M_\alpha)}$ test funkcija prostora $\mathcal{S}'^{(M_\alpha)}$ temperiranih ultradistribucija Beurlingovog tipa ([4]) je odredjen sa nekoliko ekvivalentnih familija normi, date su dve teoreme o reprezentaciji elemenata prostora $\mathcal{S}'^{(M_\alpha)}$. Ispitivane su operacije diferenciranja, ultradiferenciranja i množenja na $\mathcal{S}'^{(M_\alpha)}$ i odredjen je prostor multiplikatora prostora $\mathcal{S}'^{(M_\alpha)}$.

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