Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 23, 1 (1993), 1-3, 57 - 63

Review of Research
Faculty of Science
Mathematics Series

ON THE STABILITY OF THE FIXED POINT PROPERTY IN l_p SPACES

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Abstract

We prove, in this work, that there exists an improved constant c_p, for any $p > 1$, such that if $d(X, l_p) < c_p$, then X has the fixed point property.

AMS Mathematics Subject Classification (1991): 47H10, 46B20
Key words and phrases: fixed point property, nonexpansive mapping, normal structure.

1. Introduction

Let X be a Banach space, and K be a nonempty weakly compact convex subset of X. We will say that K has the fixed point property (in short f. p. p.) if every $T : K \to K$ nonexpansive (i. e. $\|Tx - Ty\| \leq \|x - y\|$ for every $x, y \in K$) has a fixed point, i. e. there exists $x \in K$ such that $T(x) = x$. We will say that X has f. p. p. if every weakly compact convex subset of X has f. p. p.

The fixed point property, as stated above, originated in four papers which appeared in 1965. Mainly, the presence of a geometric property, called "normal structure", implies the f. p. p. [10]. A number of abstract results were discovered, along with important discoveries related both to the structure of the fixed point sets and to techniques for approximating fixed points. The first negative result to the existence part of the theory goes to Alspach [2], who discovered an example of a weakly compact convex subset K of L^1 and an isometry $T : K \to K$ which fails to have a fixed point. This example showed that some assumption in addition to weak compactness is needed and at the same time it set the stage for Maurey's surprising discovery [12] (see also [8], [11]). For more on f. p. p., one can consult [1], [5].

2. Notations, definitions and basic facts

Let K be a nonempty weakly compact convex subset of a Banach space X. Suppose that $T : K \to K$ is nonexpansive. By Zorn's lemma, K contains a closed nonempty convex subset K_0 which is minimal for T. This means $TK_0 \subset K_0$ and no strictly smaller closed nonempty convex subset of K_0 is invariant under T. A classical argument shows that any closed nonempty convex subset of K, invariant under T, contains an approximate fixed point sequence (a. f. p. s.) (x_n), i. e. $\lim_{n\to\infty} \|x_n - Tx_n\|_X = 0$.

The following Lemma [4], [7] proved to be fundamental for the study of the f. p. p.

Lemma 1. *Suppose K_0 is a minimal weakly compact convex set for T and (x_n) is an a. f. p. s. for T. Then for all $x \in K_0$, we have*

$$\lim_{n\to\infty} \|x_n - x\| = diam(K_0).$$

Since we will be using Maurey's technique in proving our main result, let us recall some basic definitions and facts.

Definition 1. *Let X be a Banach space and let $\mathcal{U}$ be a free ultrafilter over N. The ultraproduct $\tilde{X}$ of X is the quotient space of*

$$l_\infty(X) = \{(x_n); x_n \in X \ \ for \ all \ \ n \in N \ \ and \ \ \|(x_n)\|_\infty = \sup_n \|x_n\| < \infty\},$$

by

$$\mathcal{N} = \{(x_n) \in l_\infty(X) \ \ lim_{n\to\mathcal{U}} \|x_n\| = 0\}.$$

For $(x_n) \in l_\infty(X)$, we will denote $(x_n) + \mathcal{N}$ by $(x_n)_\mathcal{U} \in \tilde{X}$. Clearly we have

$$\|(x_n)_\mathcal{U}\|_{\tilde{X}} = lim_{n\to\mathcal{U}}\|x_n\|.$$

It is also clear that X is isometric to a subspace of $\tilde{X}$ by the mapping $x \dot{\to} (x, x, ...)$. Hence, we will see X as a subspace of $\tilde{X}$ and therefore we will write $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ for the general elements of $\tilde{X}$ and x, y, z for the general elements of X.

Let K and T be as described before. Define $\tilde{K}$ and $\tilde{T}$ by

$$\tilde{K} = \{\tilde{x} \in \tilde{X};\ \text{there exists a representative } (x_n) \text{ of } \tilde{x} \text{ with } x_n \in K \ \text{ for } n \geq 1\},$$

and $\tilde{T}(\tilde{x}) = (T(x_n))_\mathcal{U}$ for any $\tilde{x} \in \tilde{K}$.

Then $\tilde{K}$ is a bounded closed convex subset of $\tilde{X}$ and $\tilde{T}(\tilde{K}) \subset \tilde{K}$. Remark that $\tilde{K}$ is not minimal for $\tilde{T}$. Indeed, let $(x_n) \subset K$ be an a. f. p. s. Then $\tilde{T}(\tilde{x}) = \tilde{x}$ where $\tilde{x} = (x_n)_\mathcal{U} \in \tilde{K}$. Recall that if $\tilde{x} = (x_n)_\mathcal{U}$ with $x_n \in K$ and $\tilde{T}(\tilde{x}) = \tilde{x}$, then there exists a subsequence $(x_{n'})$ of (x_n) which is an a. f. p. s. for T. On the other hand, let K_0 be a minimal set for T and $\tilde{x}$ be a fixed point for $\tilde{T}$ in $\tilde{K}_0$. Then for any $x \in K_0$ we have from Lemma 1

$$\|\tilde{x} - x\|_{\tilde{X}} = diam(K_0).$$

The next Lemma was proved by Maurey [12].

Lemma 2. *Suppose $\tilde{x}$ and $\tilde{y}$ are two fixed points of $\tilde{T}$ in $\tilde{K}$. Then for every $r \in (0,1)$, there exists a fixed point $\tilde{z}$ of $\tilde{T}$ so that*

$$\|\tilde{x} - \tilde{z}\| = r\|\tilde{x} - \tilde{y}\| \ \ \textit{and} \ \ \|\tilde{y} - \tilde{z}\| = (1 - r)\|\tilde{x} - \tilde{y}\|.$$

3. Main result

Let $p \in (1, \infty)$ and consider the function defined on $[0,1]$ by

$$\varphi_p(x) = \frac{1 + (1 - x)^p}{x^p + (1 - x)^p}.$$

Then $\sup_{x\in[0,1]} \varphi_p(x) = \varphi_p(x_p)$ where x_p is the only root of

$$(1 - x^{p-1})(1 - x)^{p-1} - x^{p-1} = 0$$

in $[0,1]$. It can be easily proved that

$$lim_{p\to\infty}\varphi_p(x_p) = 1, \quad lim_{p\to\infty}\varphi_p(x_p) = 2.$$

Also one can check that $x_p < \frac{1}{2^{\frac{1}{p-1}}}$.

Recall the Banach-Mazur distance between two isomorphic Banach spaces X and Y, denoted $d(X,Y)$, to be the infimum of $\|U\|\|U^{-1}\|$ taken over all bicontinuous linear operators U from X onto Y.
We now state and prove the main result of this work.

Main Theorem. *Let X be a Banach space such that*

$$d(X, l_p) < c_p = \varphi_p(x_p)^{\frac{1}{p}}$$

for some $p > 1$. Then X has f. p. p.

Proof. It is enough to prove that $X = (l_p, |\cdot|)$ has f. p. p. where $|\cdot\cdot|$ is an equivalent norm to $\|\cdot\|_p$ satisfying

$$\|\cdot\|_p \leq |\cdot| \leq d\|\cdot\|_p$$

with $d < c_p$.
Assume to the contrary that X fails to have f. p. p. Then there exist K a nonempty weakly compact convex subset and $T : K \to K$ be a nonexpansive map with no fixed point. Without any loss of generality, we can assume that K is minimal for T and $diam(K) = 1$. Classical arguments imply that K contains an a. f. p. s. (x_n) which can be assumed to be converging weakly to 0. Passing to subsequences, we may suppose that there exist coordinate projections P_{F_n} on X (with respect to the canonical Schauder basis of l_p) such that

(1) $F_n \cap F_m = \emptyset$ for $n \neq m$,

(2) $lim_{n\to\infty}|x_n - P_{F_n}(x_n)| = 0$,

(3) $lim_{n\to\infty}|x_n - x_{n+1}| = 1$.

The subsets (F_n) can be chosen to be succesive intervals and (3) holds by using Lemma 1. Put $u_n = P_{F_n}(x_n)$ for all $n \in N$. Then for $z \in l_p$ we have

$$(*) \quad \|z\|_p^p + \|z - u_n - u_{n+1}\|_p^p = \|z - u_n\|_p^p + \|z - u_{n+1}\|_p^p.$$

Let $\tilde{X}$ be an ultraproduct of X and $\tilde{K}$, $\tilde{T}$ be as defined in the previous section. Set $\tilde{x} = (x_n)_{\mathcal{U}}$ and $\tilde{y} = (x_{n+1})_{\mathcal{U}}$. Then

$$\tilde{x} = (u_n)_{\mathcal{U}} \text{ and } \tilde{y} = (u_{n+1})_{\mathcal{U}}.$$

The relation (*) translates in $\tilde{X}$ as

$$(**)\quad \|\tilde{z}\|_p^p + \|\tilde{z} - \tilde{x} - \tilde{y}\|_p^p = \|\tilde{z} - \tilde{x}\|_p^p + \|\tilde{z} - \tilde{y}\|_p^p$$

for every $\tilde{z} \in \tilde{X}$. Let $r \in (0,1)$ and $\tilde{z}$ be a fixed point of $\tilde{T}$ given by Lemma 2 such that

$$|\tilde{z} - \tilde{x}| = r \ \ and \ \ |\tilde{z} - \tilde{y}| = 1 - r.$$

Then

$$\|\tilde{z} - \tilde{x} - \tilde{y}\|_p^p \geq \frac{1}{d^p}|\tilde{z} - \tilde{x} - \tilde{y}|^p \geq \frac{1}{d^p}(1 - |\tilde{z} - \tilde{x}|)^p \geq \frac{1}{d^p}(1 - r)^p.$$

Hence,

$$\frac{1}{d^p}|\tilde{z}|^p + \frac{1}{d^p}(1-r)^p \leq \|\tilde{z}\|_p^p + \|\tilde{z} - \tilde{x} - \tilde{y}\|_p^p \leq |\tilde{z} - \tilde{x}|^p + |\tilde{z} - \tilde{y}|^p = r^p + (1-r)^p.$$

Then,

$$|\tilde{z}|^p \leq d^p(r^p + (1-r)^p) - (1-r)^p,$$

and since $|\tilde{z}| \geq \|\tilde{z}\|_p = 1$, we get $\varphi_p(r) \leq d^p$. Since r was arbitrary in $(0,1)$, we deduce that

$$\sup_{r \in (0,1)} \varphi_p(r) = \varphi_p(x_p) \leq d^p$$

which contradicts our assumption on d. The proof of the main theorem is therefore complete.

Remarks.

1. It is known [3] that if $d(X, l_p) < 2^{\frac{1}{p}}$, then X has the normal structure property and therefore via Kirk's theorem [10] has f. p. p. If $d(X, l_p) = 2^{\frac{1}{p}}$ then Bynum [3] has proved that X has f. p. p. He also gave an example of such situation where X fails to have normal structure. For $p > \frac{\ln(2)}{\ln(\frac{\sqrt{33}-3}{2})}$, one has to use Lin's result [11] to get that if $d(X, l_p) < \frac{\sqrt{33}-3}{2}$, then X has f. p. p. It is worth to mention that

$$c_p \geq c_2 = \frac{1}{\sqrt{x_2}} = (\frac{3+\sqrt{5}}{2})^{\frac{1}{2}}$$

   for every $p > 1$ and $c_2 > \frac{\sqrt{33}-3}{2}$.
   Therefore we get through the main theorem an improvement to all the well known results.

2. It is a surprising fact that the constants c_p do not decrease as p goes to ∞. To the contrary, for $p \geq 2$ the constants c_p increase to 2. Which by itself projects new light on the stability of the fixed point property (for the l_p spaces).

3. For $p = 2$ the main theorem reduces to the main result of [6].

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REZIME

O STABILNOSTI OSOBINE NEPOKRETNE TAČKE U l_p PROSTORIMA

U radu je dokazano postojanje konstante c_p, za $p > 1$, tako da ako je $d(X, l_p) < c_p$, tada X ima osobinu nepokretne tačke.

Received by the editors May 23, 1991.