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ON NUMERICAL SOLUTION OF SINGULARLY PERTURBED BOUNDARY VALUE PROBLEM III

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Abstract

A numerical solution of a semilinear singularly perturbed boundary value problem is considered. After some transformation the problem is solved by a standard difference scheme on a equidistant mesh in combination with the solution of the corresponding reduced problem. Some numerical results are given to demonstrate the efficiency of the method.

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1. Introduction

In this paper we shall consider the following singularly perturbed boundary value problem:

$$-\epsilon^2 u'' + c(x,u) = 0, \quad x \in I = [0,1], \tag{1}$$

$$u(0) = u(1) = 0,$$

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where $\epsilon \in (0, \epsilon_0), \epsilon_0 \ll 1$, is a small perturbation parameter. We assume that the following conditions are satisfied:

$$(2) \qquad c \in C^2(I \times \mathbf{R}),$$

$$0 < \gamma^2 < c_u(x,u) \leq \Gamma, \quad (x,u) \in I \times \mathbf{R}.$$

The function $c(x,u)$ and real numbers ϵ, γ, Γ are given.

Numerical treatment of problem (1) was considered among the others, in Bakhvalov [1], Boglaev [2], Herceg [6], [7], Herceg and Petrović [8], Herceg and Vulanović [9], Herceg, Vulanović and Petrović [10], Pearson [15], Petrović [16], [17], Vulanović [18], [19] and Vulanović, Herceg and Petrović [20]. This problem occurs in the study of chemical catalisys, fluid mechanics (boundary value problems), elasticity, quantum mechanics and fluid dynamics.

It is well known that under given conmditions there exists a unique solution $u_\epsilon \in C^4(I)$ to (1) which in general displays boundary layers at $x = 0$ and $x = 1$ for small ϵ, [1], [2], [5], [10], [12]. The corresponding reduced problem $c(x,u) = 0$ has also a unique solution $u_0 \in C^4(I)$ which in general does not satisfy the boundary conditions. For the solution u_ϵ of (1) it holds:

$$| u_\epsilon^{(i)}(x) | \leq \begin{cases} M(1 + \epsilon^{-i} \exp(-\gamma x/\epsilon)), & 0 \leq x \leq 0.5, \\ M(1 + \epsilon^{-i} \exp(-\gamma(1-x)/\epsilon)), & 0.5 \leq x \leq 1. \end{cases} \qquad i = 0, 1, ..., 4.$$

(3)

Here and throughout the paper M denotes any positive constant that may take a different values in different formulas, but that are always independent of ϵ and of discterization mesh.

In this paper we shall talk about a numerical solution of problem (1) which consists of $u_0(x)$ for $x \in [s, 1-s], s \in (0, 0.5)$ and numerical solutions to problems

$$(4) \qquad -\epsilon^2 v'' + c(x,v) = 0, \quad x \in [0,s],$$

$$v(0) = 0, \quad v(s) = u_0(s),$$

$$(5) \qquad -\epsilon^2 w'' + c(x,w) = 0, \quad x \in [1-s, 1],$$

$$w(1-s) = u_0(1-s), \quad w(1) = 0.$$

A choice of s is described in the section 2.

From now on we consider a numerical solution to (1) on $[0, 0.5]$. A numerical solution on $[0.5, 1]$ can be constructed in a similar way.

Let $x = \lambda(t)$, $\mu(t) = 1/\lambda'(t)$ and $w(t) = \epsilon\sqrt{\mu(t)}v(\lambda(t))$. After simple calculations we reduce (4) to

$$\text{(6)} \qquad -w'' + \frac{2\mu\mu'' - \mu'^2}{4\mu^2}w + \frac{1}{\epsilon\sqrt{\mu}\mu}c(\lambda(t), \frac{w}{\epsilon\sqrt{\mu}}) = 0,$$

$$w(0) = 0, \;\; w(\alpha) = \epsilon\sqrt{\mu(\alpha)}v(s),$$

where α is taken so that $s = \lambda(\alpha)$. If we define

$$\lambda(t) = \frac{a\epsilon t}{q-t}, \quad 0 \le t \le \alpha,$$

with $q = 0.5$, $\alpha = q - \sqrt[3]{a\epsilon}$, we have

$$s = \lambda(\alpha) = \frac{a\epsilon}{\sqrt[3]{a\epsilon}}(0.5 - \sqrt[3]{a\epsilon}).$$

Here a is an parameter chosen so that $s \in (0, 0.5)$, i.e. $a\epsilon < 1/8$.

It is easy to calculate that

$$\lambda'(t) = \frac{a\epsilon q}{(q-t)^2},$$

$$a\epsilon q\mu(t) = (q-t)^2, \; a\epsilon q\mu'(t) = -2(q-t), \; a\epsilon q\mu''(t) = 2,$$
$$2\mu''(t)\mu(t) - \mu'(t)^2 = 0.$$

Now, from (6) we obtain

$$\text{(7)} \qquad -w'' + \frac{aq\sqrt{aq\epsilon}}{(q-t)^3}\, c(\lambda(t), \sqrt{\frac{aq}{\epsilon}}\frac{w(t)}{q-t}) = 0,$$

$$w(0) = 0, \quad w(\alpha) = \epsilon\sqrt{\mu(\alpha)}u_0(s).$$

Let $I_s = \{t_i = ih : i = 0, 1, ..., n\}$ be the discretazition mesh with $n \in \mathrm{N}$, $h = \alpha/n$. On I_s we solve (7) numerically using a second order difference scheme. If we denote this solution by $w^h = [w_0, w_1, ..., w_n]^T$, than

$$| \, w(t_i) - w_i \, | \le Mh^2, \quad i = 0, 1, ..., n,$$

where $w(t)$ is exact solution of (7).

Since $w(t) = \epsilon\sqrt{\mu(t)}v(\lambda(t))$ we define the numerical solution $v^h = [v_0, v_1, ..., v_n]^T$ to (4) as

$$v_i = \frac{1}{\epsilon\sqrt{\mu(t_i)}} w_i, \quad i = 0, 1, ..., n.$$

So, we obtain v_i as the numerical solution of the problem (1) at points $\lambda(ih)$, $i = 0, 1, ..., n$. The existence of $v(t)$ and the following estimate

$$(6) \qquad | u_\epsilon(x) - v(x) | \leq M(\exp(-\gamma s/\epsilon) + \epsilon^2), \quad x \in [0, s],$$

follow from the inverse monotonicity of (4) under assumption (2), see [12], [18]. For $s \leq x \leq 0.5$ we approximate $u_\epsilon(x)$ by $u_0(x)$ and it holds

$$(7) \qquad | u_\epsilon(x) - u_0(x) | \leq M(\exp(-s\gamma/\epsilon) + \epsilon^2), \quad x \in [s, 0.5].$$

Using (9) and (10) we prove

$$| u_\epsilon(x) - u(x) | \leq M(h^2 + \epsilon^2), \qquad x \in I_s \cup [s, 0.5],$$

where

$$(8) \qquad u(x) = \begin{cases} v_i & \text{for} \quad x = x_i \in I_\lambda, \\ u_0(x) & \text{for} \quad x \in [s, 0.5], \end{cases}$$

where

$$I_\lambda = \{\lambda(ih) : i = 0, 1, ..., n\}.$$

Our numerical results are obtained by solving boundary value problems which were considered in many papers: [3], [5], [19], [20]. These results show that the theoretical order of convergence is also established numerically.

2. The numerical method

We use the discretization mech from the previous section, i.e. on the interval $[0, \alpha]$ we place the uniform mesh

$$t_i = ih, \quad i = 0, 1, ..., n, \quad h = \frac{\alpha}{n}.$$

To approximate $w(t)$ on this mesh we define a mesh function $w^h = [w_0, w_1, ..., w_n]^T$ as the solution of a system of finite - difference equations

$$(12) \qquad -h^{-2}(w_{j+1} - 2w_j + w_{j+1}) + f(t_j, w_j) = 0,\ j = 1, 2, ..., n-1,$$

$$w_0 = 0, \quad w_n = \epsilon\sqrt{\mu(\alpha)}u_0(s),$$

where

$$f(t,x) = \frac{aq\sqrt{aq\epsilon}}{(q-t)^3}c(\lambda(t), \sqrt{\frac{aq}{\epsilon}}\frac{x}{q-t}).$$

The resulting difference equations (12) are in general nonlinear and we shall emplay iterative methods to solve them. Since

$$\frac{\partial f}{\partial x}(t,x) = \frac{aq\sqrt{aq\epsilon}}{(q-t)^3} \cdot \frac{\partial c}{\partial x}(t,x) \cdot \frac{1}{q-t} \cdot \sqrt{\frac{aq}{\epsilon}},$$

we have

$$0 < Q_* \leq \frac{\partial f}{\partial x}(t,x) \leq Q^*,$$

where

$$Q_* = \frac{a^2\gamma^2}{q^2}, \quad Q^* = \frac{a^2q^2\gamma^2}{(a\epsilon)^{3/4}}.$$

Now, in order to solve (12), we can apply the functional iteration scheme or Newton's method from [11].

Theorem 1. *Suppose that conditions (2) and (3) are satisfied. Then the numerical solution w^h of the difference equations (12) and the solution $w(t)$ of the boundary value problem (7) satisfy*

$$| w(t_j) - w_j | \leq Mh^2, \ \ j = 0, 1, ..., n.$$

Proof. In the same way as in [11] we can prove that

$$| w(t_j) - w_j | \leq M \max_{1 \leq i \leq n-1} | \tau_j(w) |, \quad j = 0, 1, ..., n$$

where

$$\tau_j(w) = -\frac{h^2}{12}w^{iv}(\theta_j), \quad \theta_j \in (t_{j-1}, t_j), \ \ j = 1, 2, ..., n-1.$$

To end of our proof we need prove

$$| w^{iv}(t) | \leq M, \quad t \in [0,\alpha].$$

From

$$w(t) = \epsilon\sqrt{\mu(t)}v(\lambda(t))$$

we obtain

$$w^{iv}(t) = \frac{\epsilon^3 a^2 q^2}{\sqrt{a\epsilon q}}\left(\frac{12}{(q-t)^5}v''(\lambda(t)) + \frac{8a\epsilon q}{(q-t)^6}w'''(\lambda(t)) + \frac{a^2\epsilon^2 q^2}{(q-t)^7}v^{iv}(\lambda(t))\right).$$

Since for $i = 0,1,2,3,4$

$$| v^{(i)}(t) | \leq M(1+\epsilon^{-i}exp(\frac{\gamma}{\epsilon})), \ t \in [0,0.5],$$

we have

(13) $$| w^{iv}(t) | \leq M\frac{\epsilon}{\sqrt{a\epsilon q}}a^2\epsilon^2 q^2\left[\frac{12}{(q-t)^5} + \frac{8a\epsilon q}{(q-t)^6} + \frac{a^2\epsilon^2 q^2}{(q-t)^7} + \frac{1}{\epsilon^2}(12z_5(t) + 8aqz_6(t) + a^2q^2z_7(t))\right],$$

with

$$z_k(t) = \frac{exp(-\frac{\gamma a t}{q-t})}{(q-t)^k}, \quad k \in \mathbf{N}.$$

It is easy to show

$$z_k(t) \leq z_{k+1}(t), \quad t \in [0,\alpha], \ k \in \mathbf{N},$$

$z'_k(t) = 0$ if and only if $k(q-t) = aq\gamma$. So, we have for $t \in [0,\alpha]$

$$0 < z_k(t) \leq z_k(q - \frac{aq\gamma}{k}) = \frac{e^{a\gamma - k}}{(\frac{aq\gamma}{k})^k}, \ k \in \mathbf{N},$$

i.e.

(14) $$z_k(t) \leq M, \quad t \in [0,\alpha], \quad k \in \mathbf{N}.$$

From

$$q - t \geq q - \alpha = \sqrt[3]{a\epsilon}$$

it follows

$$\frac{1}{(q-t)^k} \leq (a\epsilon)^{-k/3}, \quad k \in \mathbf{N}, \tag{15}$$

and

$$\frac{1}{(q-t)^k} \leq \frac{a\epsilon}{(q-t)^{k+1}}, \quad k \in \mathbf{N}.$$

Using this we conclude

$$| w^{iv}(t) | \leq M\sqrt{\frac{\epsilon}{aq}} a^2q^2[12+8q+q^2)\frac{\epsilon^2}{(q-t)^5}+$$

$$+(12+8aq+a^2q^2)z_7(t)],$$

$$| w^{iv}(t) | \leq M\sqrt{\frac{\epsilon}{aq}}(\frac{\epsilon^2}{(q-t)^5}+z_7(t)), \tag{16}$$

$$| w^{iv}(t) | \leq M\sqrt{\frac{\epsilon}{aq}}(\epsilon^{1/3}+M) \leq M.$$

Theorem 2. *Let us suppose the conditions (2) and (3) are satisfied. Then the numerical solution* $v^h = [v_0, v_1, ..., v_n]^T$, *with*

$$v_i = \sqrt{\frac{aq}{\epsilon}}\frac{w_i}{q-t_i}, \quad i = 0, 1, ..., n$$

and exact solution $v(t)$ *of the boundary value problem (4) satisfy*

$$| v(\lambda(t_i)) - v_i | \leq Mh^2.$$

Proof. As in proof of Theorem 1 we have

$$| w(t_j) - w_j | \leq h^2 \max_{t\in[0,\alpha]} | w^{iv}(t) | .$$

Using (14), (15) and (16) we obtain

$$| \sqrt{\frac{aq}{\epsilon}}\frac{1}{q-t_j}w(t_j) - \sqrt{\frac{aq}{\epsilon}}\frac{w_j}{q-t_j} | \leq h^2M(\frac{\epsilon^{1/3}}{q-t_j}+z_8),$$

i.e.

$$| v(\lambda(t_j)) - v_j | \leq Mh^2. \quad \square$$

Theorem 3. *Suppose the conditions (2) and (3) are satisfied. Let u_ϵ be the solution to problem (1) and let u be defined by (11). Then holds*

$$| u_\epsilon(t) - u(t) | \leq M(h^2 + \epsilon^2), \quad t \in I_\lambda \cup [s, 0.5],$$

where

$$I_\lambda = \{\lambda(ih) : i = 0, 1, ..., n\}.$$

Proof. From (9) it follows that (10) is valid for $t \in [s, 0.5]$. For $t = \lambda(ih)$ we have

$$| u_\epsilon(t) - u(t) | \leq | u_\epsilon(t) - v(t) | + | v(t) - v_i | .$$

Since, (8),

$$| u_\epsilon(t) - v(t) | \leq M(\exp(-s\gamma/\epsilon) + \epsilon^2), \quad t \in [0, s],$$

and

$$s = 0.5a\epsilon/\sqrt[3]{a\epsilon} - a\epsilon,$$

it can be shown

$$exp(-\frac{s\gamma}{\epsilon}) \leq M\epsilon^2,$$

$$| u_\epsilon(t) - v(t) | \leq M\epsilon^2.$$

From this and Theorem 2 it follows our statement. □

3. The numerical example

In this section we present the results of numerical experiments using the scheme described in previous section. Our example

$$-\epsilon^2 u'' + u + \cos^2(\pi x) + 2(\epsilon\pi)^2 \cos(2\pi x) = 0,$$

$$u(0) = u(1) = 0,$$

is often used in the literature to compare different codes. We also give the numerical validation of the theoretical order of convergence for the scheme discused in section 2.

The solution of our problem is

$$u_\epsilon(x) = \frac{\exp(-x/\epsilon) + \exp(-(1-x)/\epsilon)}{1 + \exp(-1/\epsilon)} - \cos^2(\pi x),$$

with $\gamma = 1$ and $\Gamma = 1$.

We denote by E_n the maximum of $| u_\epsilon(x) - u(x) |$, $x \in I_\lambda$, i.e.

$$E_n = \max\{| u_\epsilon(x) - u(x) |: \quad i = 0, 1, ..., n\}.$$

Also, we define in the usual way the order of convergence Ord for two successive values of n with respective errors E_n and E_{2n} :

$$Ord = \frac{\log E_n - \log E_{2n}}{\log 2}.$$

We expect that $Ord = 2$ for small ϵ. In our calculation $a = 1$. Table 1 contains E_n and Ord for $n = 2^k$, $k = 2, 3, ..., 10$. For ϵ we take $2^{-m}, m = 9, 10, ..., 20$. In the Table 2 we present α for different values of ϵ.

Table 1.

$n \backslash \epsilon$	2^{-10}	2^{-12}	2^{-14}	2^{-16}	2^{-18}	2^{-20}
4	1.760 (-2) -	1.301 (-2) -	1.934 (-2) -	2.255 (-2) -	2.396 (-2) -	2.455 (-2) -
8	1.760 (-2) 0	3.150 (-3) 2.046	2.942 (-3) 2.717	3.426 (-3) 2.719	3.771 (-3) 2.668	3.987 (-3) 2.622
16	1.760 (-2) 0	9.119 (-4) 1.788	7.209 (-4) 2.029	7.636 (-4) 2.166	8.086 (-4) 2.221	8.315 (-4) 2.262
32	1.760 (-2) 0	9.119 (-4) 0	1.702 (-4) 2.083	1.798 (-4) 2.087	1.886 (-4) 2.100	1.924 (-4) 2.111
64	1.760 (-2) 0	9.119 (-4) 0	4.120 (-5) 2.047	4.381 (-5) 2.037	4.554 (-5) 2.050	4.664 (-5) 2.045
128	1.760 (-2) 0	9.119 (-4) 0	1.014 (-5) 2.023	1.079 (-5) 2.022	1.121 (-5) 2.023	1.148 (-5) 2.023
256	1.760 (-2) 0	9.119 (-4) 0	8.300 (-6) 0.288	2.676 (-6) 2.011	2.780 (-6) 2.011	2.847 (-6) 2.011
512	1.760 (-2) 0	9.119 (-4) 0	8.300 (-6) 0	6.663 (-7) 2.006	6.923 (-7) 2.006	7.089 (-7) 2.006
1024	1.760 (-2) 0	9.119 (-4) 0	8.300 (-6) 0	1.662 (-7) 2.003	1.727 (-7) 2.003	1.769 (-7) 2.003

Table 2.

ϵ	2^{-10}	2^{-12}	2^{-14}	2^{-16}	2^{-18}	2^{-20}
α	0.407870	0.437500	0.46627	0.475197	0.484375	0.490157

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REZIME

O NUMERIČKOM REŠAVANJU SINGULARNO PERTURBOVANOG KONTURNOG PROBLEMA III

Numerički se rešava semilinearni singularno perturbovani konturni problem. Posle odredjenih transformacija problem se rešava primenom standardne diferencne šeme na ekvidistantnoj mreži u kombinaciji sa redukovanim rešenjem. Numerički primeri ilustruju efikasnost predloženog postupka i potvrdjuju teoretske rezultate.

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