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ON 3-CYCLES IN n-PARTITE TOURNAMENTS

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Abstract

We prove that every strong n-partite ($n \geq 3$) tournament has at least $(n-2)$ 3-cycles. This generalizes the results of [1] and [2] corresponding to ordinary and 3-partite tournaments.

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An $n-$*partite tournament* $(n \geq 2)$ is obtained from a complete $n-$partite graph orienting all its edges. An ordinary tournament of order n is an $n-$partite tournament whose each partite set is a single vertex. It is known from [1] that each strong ordinary tournament of order n contains at least $(n-2)$ 3-cycles (cycles of length 3). The same result has been obtained for strong 3-partite tournaments in [2]. We prove that the same holds for strong $n-$partite tournaments $(n \geq 3)$.

Basic definitions and terminology will follow those of [1] and [2] .

Lemma 1. *If C is a longest cycle of a strong n-partite tournament $T(V_1, ..., V_n)$ $(n \geq 2)$ then C meets each partite set, i.e. $V(C) \cap V_i \neq \emptyset$ for each $i = 1, ..., n$.*

Proof. The case $n = 2$ is trivial. So, assume that $n \geq 3$. Suppose that $C : x_1 ... x_m x_1$ is a longest cycle of $T(V_1, ..., V_n)$ which avoids a partite set V_i, $i \in \{1, ..., n\}$. Let v be any vertex of V_i. If v dominates and is dominated by vertices of C, then there exist $x_i, x_{i+1} \in V(C)(i = 1, ..., m$; indices are taken modulo m) such that $x_i \to v$ and $v \to x_{i+1}$. Thus, v can be inserted in C between x_i and x_{i+1} producing a longer cycle $x_1 ... x_i v x_{i+1} ... x_m x_1$. This implies that v dominates or is dominated by all vertices of C. Assume the former. Since T is strong there is a path from C to v. Let it be $x_j z_1 ... z_\ell v$, $j \in \{1, ..., m\}$, $\ell \geq 1$, $z_i \notin V(C)$, $i = 1, ..., \ell$. But now $v x_{j+1} ... x_n x_1 ... x_j z_1 ... z_\ell v$ is a cycle longer that C; a contradiction . □

Lemma 2. *Let $C : x_1 ... x_m x_1$ be a cycle of a strong $n-$partite tournament $T(V_1, ..., V_n)(n \geq 3)$ such that $V(C) \cap V_i \neq \emptyset$ for each $i = 1, ..., n$. Then there are at least $n - 1$ triples $(x_i, x_{i+1}, x_{i+2})(i = 1, ..., m$; indices are taken modulo m) such that $x_i \in V_j$, $x_{i+1} \in V_k$, $x_{i+2} \in V_\ell$, $j \neq k$, $k \neq \ell$, $\ell \neq j$.*

Proof. Without loss of generality we may assume that $x_1 \in V_1$, $x_2 \in V_2$, the first vertex along C from a new set belongs to V_3 and the next one to V_4 etc. If x_i is the first vertex from V_3 comming after x_2 then (x_{i-2}, x_{i-1}, x_i) is a desired triple. Arguing in the same way we conclude that (x_{j-2}, x_{j-1}, x_j) in $(n-2)$th triple where x_j is the first appearance of a vertex from V_n. Now, we consider the first appearance of a vertex from V_1 after x_j. If it is $x_{j+s}(s > 1)$ then $(x_{j+s-2}, x_{j+s-1}, x_{j+s})$ is $(n-1)$ th triple. If it is x_{j+1} and $x_{j-1} \notin V_j$ the triple is (x_{j-1}, x_j, x_{j+1}). Finally, if $x_{j-1} \in V_1$ we look for the first appearance of a vertex of V_2 after x_{j+1}. Let it be x_p. Then the wanted triple is (x_{p-2}, x_{p-1}, x_p). It is routine to check that all described triples are distinct. □

Theorem 1. *If $T(V_1, ..., V_n)(n \geq 3)$ is a strong n-partite tournament, then T contains at least $(n-2)$ $3-$cycles.*

Proof. By induction on n and $m = \sum_{i=1}^{n} | V_i | \geq n$. Let $C : x_1 ... x_k x_1$ be a longest cycle of T. By Lemma 1 C meets each $V_i(i = 1, ..., n)$ implying $k \geq n$. According to Lemma 2 there are at least $n - 1$ triples $(x_{i_1}, x_{i_1+1}, x_{i_1+2}), ...,$ $(x_{i_{n-1}}; x_{i_{n-1}+1}, x_{i_{n-1}+2})(\{i_1, ..., i_{n-1}\} \subset \{1, ..., k\}$; indices are taken modulo k). If $n - 2$ of them are cyclic the proof is finished. So, assume that at least 2 triples are transitive. Let $(x_{i_1}, x_{i_1+1}, x_{i_1+2})$ is one of them, where $x_{i_1} \in V_j$. Denote by T_1 the tournament induced by cycle $C_1 : x_1 ... x_{i_1} x_{i_1+2} ... x_k x_1$ of length $k - 1$.

(a) $| V(C) \cap V_j | > 1$. Then C_1 meets all partite sets $V_1, ..., V_n$ and T_1 is also a strongly connected $n-$partite tournament on $< m$ vertices. By induction hypothesis it contains at least $(n-2)$ 3 - cycles. So, does T.

(b) $| V(C) \cap V_j | = 1$. It means that C_1 avoids $V_j.T_1$ is a strong $(n-1)$-partite tournament. By induction hypothesis T_1 contains at least $(n-3)$ 3 - cycles. Consider now the vertex $x_{i_1+1} = u$. If u dominates and is dominated by some vertices of C_1 there are two consecutive vertices y_i and y_{i+1} on C such that $y_i \to u$, $y_i \to y_{i+1}, u \to y_{i+1}$. If not u dominates (or is dominated by) all vertices of C_1. Then there is a path $y_i z_1 ... z_2 u$ where $y_i \in V(C_1)$, $z_i \notin V(C_1)$, $i = 1, ..., q$, $(q \geq 1)$, as T is strong. As $u \to y_i$ and $z_q \to u$ there are z_{r-1} and z_r, $r \in \{1, ..., q\}$, $(z_0 = y_i)$ such that $vz_{r-1}z_rv$ is a 3-cycle. Anyway we meet at least one new 3-cycle. So, the total number is at least $n-2$. □

This result cannot be improved. In [1] a strong ordinary tournament was given having precisely $(n-2)$ 3-cycles.

References

[1] Harary, F., Moser, L., The theory of round robin tournaments, Amer. Math. Monthly 73 (1966), 231-246.

[2] Godard, W. and all , On multipartite tournaments, Journal of Comb. Theory, Series B52, 284-300 (1991).

REZIME

KONTURE DUŽINE 3 U n- PARTITNIM TURNIRIMA

Pokazano je da svaki jako povezan n - partitni ($n \geq 3$) turnir sadrži bar $n-2$ kontura dužine 3. To je generalizacija rezultata F. Harary-ja i W. Godard-a koji se odnose na obične i 3 - partitne turnire.

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