Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 23, 1 (1993), 269 - 282

Review of Research
Faculty of Science
Mathematics Series

ON THE CONDITIONAL CAUCHY EQUATION ON 3-ADIC GROUPS

Antoni Chronowski
Wyzsza Szkola Pedagogiczna, Institut Matematyki
ul. Podchorazych 2, 30-084 Krakow, Poland

Abstract

The paper deals with the general solutions of the Pexider equation and Cauchy equation on 3-adic groups. Moreover, the characterization of the Cauchy i-nuclei is presented.

AMS Mathematics Subject Clasiffication (1991): 39B50, 20N15.
Key words phrases: n-group, functional equation, Pexider equation, Cauchy equation.

This paper is concerned with the general solutions of two types of the conditional Cauchy equation on 3-adic groups. Moreover, the general solution of the Pexider equation 3-adic groups is presented. The paper also contains the characterization of the algebraic and set-theoretical structure of the Cauchy i-nuclei for a function defined on n-adic groups.

The definitions, theorems, and notations related to the n-adic group theory are based on papers [1], [3], [4], [6], [7].

We begin with the definition of the Cauchy i-nucleus for a function defined on n-adic groups.

Let $A()$ and $B[\,]$ be n-adic groups. Let $f : A \to B$ be any function. For every $i = 1, 2, ..., n$ we define the set

$$N_f^i = \{x \in A : \forall x_1, ..., x_{i-1}, x_{i+1}, ..., x_n \in A :$$

$f((x_1, ..., x_{i-1}, x, x_{i+1}, ..., x_n)) = [f(x_1), ..., f(x_{i-1}), f(x), f(x_{i+1}), ..., f(x_n)]\}$.
The set N_f^i (for $i = 1, 2, ..., n$) is said to be the Cauchy i-nucleus for the function f (cf. [5], p. 482).

Notice that if there exists an $i_0 \in \{1, 2, ..., n\}$ such that $N_f^{i_0} = A$, then $N_f^i = A$ for every $i \in \{1, 2, ..., n\}$.

To examine the problem of the algebraic and set - theoretical structure of the Cauchy i-nuclei we distinguish three cases: $n = 2$, $n = 3$, $n > 3$.

If $n = 2$ the Cauchy i-nuclei for a function defined on groups are empty sets or subgroups (cf. [5], p. 482).

We shall first deal with the case of $n > 3$.

Theorem 1. *Let $A()$ and $B[\,]$ be n-adic groups for $n > 3$. Let $f : A \to B$ be any function. Then $N_f^i = \emptyset$ or $N_f^i = A$ for every $i \in \{1, 2, ..., n\}$.*

Proof. Suppose that $N_f^i \neq \emptyset$ for any arbitrary fixed $i \in \{1, 2, ..., n\}$. Let us distinguish the following two cases.

(a) The case for $1 \leq i < n - 1$.
If $x \in A$ then $x = (a_1^{n-1}, y)$ for the arbitrary fixed elements $a_1, a_2, ..., a_{n-1} \in N_f^i$ and for a certain element $y \in A$. Thus

$$f((x_1^{i-1}, x, x_{i+1}^n)) = f((x_1^{i-1}, (a_1^{n-1}, y), x_{i+1}^n))$$
$$= f((x_1^{i-1}, a_1, (a_2^{n-1}, y, x_{i+1}), x_{i+2}^n))$$
$$= [f(x_1), ..., f(x_{i-1}), f(a_1), [f(a_2), ...,$$
$$f(a_{n-1}), f(y), f(x_{i+1})], f(x_{i+2}), ..., f(x_n)]$$
$$= [f(x_1), ..., f(x_{i-1}), [f(a_1), f(a_2), ... f(a_{n-1}), f(y)], f(x_{i+1}), ..., f(x_n)]$$
$$= [f(x_1, ..., f(x_{i-1}), f(x), f(x_{i+1}), ..., f(x_n)]$$

for all $x_1, ..., x_{i-1}, x_{i+1}, ..., x_n \in A$.

(b) The case for $i = n - 1$ or $i = n$.

If $x \in A$ then $x = (y, a_2^n)$ for the arbitrary fixed elements $a_1, ..., a_n \in N_f^i$ and for a certain element $y \in A$.

To check that case (b) is valid it is enough to use a similar calculation technique as in case (a).

Corollary 1. *Let $A()$ and $B[\,]$ be n-indic groups for $n > 3$. Let $f : A \to B$ be any function. If there exists an $i_0 \in \{1, 2, ..., n\}$ such that $N_f^{i_0} \neq \emptyset$, then $N_f^i = A$ for every $i \in \{1, 2, ..., n\}$.*

Now we pass on to the case for $n = 3$. The following statements will be found useful in the proof of the next theorem.

Let $A()$ be a 3-adic group. The skew element for the element $a \in A$ will be denoted by $\bar{a}$. Let us notice that $\overline{(a,a,a)} = (\bar{a}, \bar{a}, \bar{a})$ for every $a \in A$. Indeed, it is easy to check that $((\bar{a}, \bar{a}, \bar{a}), (a, a, a), (a, a, a)) = (a, a, a)$ for an arbitrary $a \in A$.

Let $A()$ and $B[\,]$ be 3-adic groups. Suppose that for a function $f : A \to B$ the set N_f^2 is non-empty. Then $f(\bar{a}) = \overline{f(a)}$ for every $a \in N_f^2$. Indeed, since $\bar{a} = (\bar{a}, a, \bar{a})$, we have $f(\bar{a}) = f((\bar{a}, a, \bar{a})) = [f(\bar{a}), f(a), f(\bar{a})]$. On the other hand, $[\overline{f(a)}, f(a), f(\bar{a})] = f(\bar{a})$ and consequently $f(\bar{a}) = \overline{f(a)}$ for every $a \in N_f^2$.

Theorem 2. *Let $A()$ and $B[\,]$ ne $3-$adic groups. Let $f : A \to B$ be any function. Then*

(a) $N_f^1 = \emptyset$ or $N_f^1 = A$,

(b) $N_f^2 = \emptyset$ or N_f^2 is a $3-$adic subgroup of the $3-$adic group $A()$,

(c) $N_f^3 = \emptyset$ or $N_f^3 = A$.

Proof. The prove conditions (a) and (c) it is enough to imitate the proof of Theorem 1. Let us consider condition (b). Assume that $N_f^2 \neq \emptyset$ and $a_1, a_2, a_3 \in N_f^2$. Then

$$\begin{aligned} f((x_1, (a_1, a_2, a_3), x_3)) &= f(((x_1, a_1, a_2), a_3, x_3)) \\ &= [[f(x_1), f(a_1), f(a_2),], f(a_3), f(x_3)] \\ &= [f(x_1), [f(a_1), f(a_2), f(a_3)], f(x_3), \\ &= [f(x_1), f((a_1, a_2, a_3)), f(x_3)] \quad \text{for all } x_1, x_3 \in A. \end{aligned}$$

Thus $(a_1, a_2, a_3) \in N_f^2$ for all $a_1, a_2, a_3 \in N_f^2$.

Taking an arbitrary element $a \in N_f^2$ we shall prove that $\bar{a} \in N_f^2$. Indeed,

$$f((x_1, \bar{a}, x_3)) = f((x_1, (\bar{a}, \bar{a}, a), x_3)) = f(((x_1, \bar{a}, \bar{a}), a, x_3))$$

$$\begin{aligned}
&= [f((x_1,\bar{a},\bar{a})), f(a), f(x_3)] = [f((x_1),(a,\bar{a},\bar{a}),\bar{a})), f(a), f(x_3)] \\
&= [f((x_1), a, (\bar{a},\bar{a},\bar{a}))), f(a), f(x_3)] \\
&= [[f(x_1), f(a), f((\bar{a},\bar{a},\bar{a}))], f(a), f(x_3)] \\
&= [[f(x_1), f(a), [\overline{f(a)}, \overline{f(a)}, \overline{f(a)}]], f(a), f(x_3)] \\
&= [[f(x_1), [f(a), \overline{f(a)}, \overline{f(a)}], \overline{f(a)}], f(a), f(x_3)] \\
&= [[f(x_1), \overline{f(a)}, \overline{f(a)}, f(a), f(x_3)] \\
&= [f(x_1), [\overline{f(a)}, \overline{f(a)}, f(a)], f(x_3)] \\
&= [f(x_1), \overline{f(a)}, f(x_3)] = [f(x_1), f(\bar{a}), f(x_3)] \quad \text{for all } x_1, x_3 \in A.
\end{aligned}$$

We shall construct an example of a 3-adic group $A()$ and a function $f : A \to A$ for which $\emptyset \neq N_f^2 \neq A$.

Example 1. Let us consider the Klein group $A = \{e, a, b, c\}$ under the operation:

	e	a	b	c
e	e	a	b	c
a	a’	e	c	b
b	b	c	e	a
c	c	b	a	e

The function $\alpha : A \to A$ is defined by setting: $\alpha(e) = e$, $\alpha(a) = b$, $\alpha(b) = a$, $\alpha(c) = c$. The function α is an automorphism of the Klein group A. The 3-ary operation on the set A is defined as follows

$$(x_1, x_2, x_3) = x_1 \alpha(x_2) x_3$$

for all $x_1, x_2, x_3 \in A$.

It is easy to verify that $A()$ forms a 3-adic group. Next, the function $f : A \to A$ is defined by putting: $f(e) = e$, $f(a) = c$, $f(b) = a$, $f(c) = b$. The function f is an automorphism of the Klein group. We shall show that $N_f^2 = \{e\}$. Indeed, $f((x_1, e, x_3)) = f(x_1 x_3) = f(x_1) f(x_3)$ and $(f(x_1), f(e), f(x_3)) = (f(x_1), e, f(x_3)) = f(x_1) f(x_3)$ for all $x_1, x_3 \in A$. Suppose that $a \in N_f^2$. Then $f((b, a, c)) = f(bbc) = f(c) = b$ and $(f(b), f(a), f(c)) = (a, c, b) = acb = e$. Thus we have obtained a contradiction. Similarly, we can check that $b \notin N_f^2$ and $c \notin N_f^2$. Let us notice that $N_f^1 = N_f^3 = \emptyset$.

It is easy to verify the following

Remark 1. If 3-adic groups $A()$ and $B[\ ]$ are commutative and $N_f^2 \neq \emptyset$, then $N_f^2 = A$ for an arbitrary function $f : A \to B$.

In the sequel we shall use the retracts for $n-$ adic groups (cf. [3], [4]).

According to the Hosszú theorem (cf. [4]) for an arbitrary $n-$ adic group $A()$ there exists a binary group $(A, \cdot)$, an automotphism $\alpha \in Aut(A, \cdot)$, and an element $a \in A$ such that $\alpha(a) = a$, $\alpha^{n-1}(x) = axa^{-1}$ for every $x \in A$, and $(x_1, x_2, x_3, ..., x_{n-1}, x_n) = x_1 \cdot \alpha(x_2) \cdot \alpha^2(x_3) \cdot ... \cdot \alpha^{n-2}(x_{n-1})$ $\cdot a \cdot x_n$ for all $x_1, x_2, x_3, ..., x_{n-1}, x_n \in A$.

The system $(A, \cdot, \alpha, a)$ is said to be a binary retract of the $n-$adic group $A()$ (cf. [3]).

For the sake of simplicity we shall call a binary retract a retract an often treat it as a group. Instead of $(A, \cdot, \alpha, a)$ we shall also write (A, α, a).

The retract can be used for the construction of $n-$adic groups (cf. [4]). It is easy to verify that if $(A, \cdot, \alpha, a)$ and $(A, \cdot, \alpha_1, a_1)$ are retract (with the same operation $\cdot$) of an $n-$adic group $A()$, than $\alpha = \alpha_1$ and $a = a_1$.

Sokolov (cf. [7]) gives a very useful method of constructing a retract $(A, \circ, \alpha, a)$ for an $n-$adic group $A()$.

Namely.

$$x \circ y = (x, p^{n-2}, y),$$

$$\alpha(x) = (\bar{p}, x, p^{n-2}),$$

$$a = (\bar{p}^n)$$

for an arbitrary fixed element $p \in A$ and for all $x, y \in A$. The set A with the operation $\circ$ forms a group for which $\bar{p}$ is an indentity.

We shall present a few remarks on the Sokolov method of constructing retract for 3-adic groups.

Let (A, α, a) be an arbitrary fixed retract of a 3-adic group $A()$. We shall prove that the retract (A, α, a) can be constructed by means of the Sokolov method. Since $\alpha(a) = a$, we have $\alpha(a^{-1}) = a^{-1}$. Thus $x \circ y = (x, a^{-1}, y) = x\alpha(a^{-1})ay = xy$ for all $x, y \in A$.

If a 3-adic group $A()$ is commutative, then its every retract is of the form (A, id_A, a). Indeed, suppose that (A, α, a) is a retract of the 3-adic group $A()$. Then $\alpha(x) = (\overline{a^{-1}}, x, a^{-1}) = (\overline{a^{-1}}, a^{-1}, x) = x$ for every $x \in A$.

Now we pass on to the definition of the Pexider equation on $n-$adic groups.

The Pexider equation on $n-$adic groups $A()$ and $B[]$ is said to be the following functional equation

$$\alpha_{n+1}((x_1, x_2, ..., x_n)) = [\alpha_1(x_1), \alpha_2(x_2), ..., \alpha_n(x_n)]$$

for all $x_1, x_2, ..., x_n \in A$, where $\alpha_1, \alpha_2, ..., \alpha_{n+1} : A \to B$ are unknown functions.

First, we shall consider a certain functional equation on groups and the obtained results will be used in the proof of the theorem on the general solution of the Pexider equation on 3-adic groups.

Let us consider on groups A and B the following functional equation

$$\alpha_3(\mu_1(x_1)\mu_2(x_2)) = \alpha_1(x_1)\alpha_2(x_2) \tag{1}$$

for all $x_1, x_2 \in A$, where $\mu_1, \mu_2 : A \to A$ are given bijections and $\alpha_1, \alpha_2, \alpha_3 : A \to B$ are unknown functions.

Denote by L_{a_1} and R_{a_2} the left translation and the right translation, respectiveluy.

Theorem 3. *If a triple of functions $(\alpha_1, \alpha_2, \alpha_3)$ is the solution of equation (1) on groups A and B, then there exists a homomorphism $\varphi : A \to B$ of the groups A and B, and elements $a_1, a_2 \in B$ such that*

$$\alpha_1 = L_{a_1}\varphi\mu_1, \quad \alpha_2 = R_{a_2}\varphi\mu_2, \quad \alpha_3 = L_{1_1}R_{a_2}\varphi. \tag{2}$$

If $\varphi : A \to B$ is a homomorphism of groups A and B, and $a_1, a_2 \in B$ are arbitrary elements, then a triple of functions $(\alpha_1, \alpha_2, \alpha_3)$ of form (2) is the solution of equation (1).

Proof. Let a triple of functions $(\alpha_1, \alpha_2, \alpha_3)$ be the solution of equation (1).

Since μ_1 and μ_2 are bijections on the set A, equation (1) can be written in the equivalent form

$$\alpha_3(x_1x_2) = (\alpha_1\mu_1^{-1})(x_1)(\alpha_2\mu_2^{-1})(x_2)$$

for all $x_1, x_2 \in A$.

Let us put $\beta_1 = \alpha_1\mu_1^{-1}$ and $\beta_2 = \alpha_2\mu_2^{-1}$. Consequently $\alpha_3(x_1x_2) = \beta_1(x_1)\beta_2(x_2)$ for all $x_1, x_2 \in A$. Put $a_1 = \beta_1(1)$ and $a_2 = \beta_2(1)$. Define the function $\varphi(x) = a_1^{-1}\beta_1(x)$ for every $x \in A$. Notice that $\alpha_3(x) = \beta_1(1)\beta_2(x) = a_1\beta_2(x)$ and $\alpha_3(x) = \beta_1(x)\beta_2(1) = \beta_1(x)a_2$ for every $x \in A$. Hence $\alpha_3(x) = \beta_1(x)a_2 = a_1(a_1^{-1}\beta_1(x))a_2 = a_1\varphi(x)a_2$ for every $x \in A$. Thus $\beta_1(x) = a_1\varphi(x), \beta_2(x) = \varphi(x)a_2, \alpha_3(x) = a_1\varphi(x)a_2$ for every $x \in A$. It is easy to check that $\varphi : A \to B$ is a homomorphism of groups A and B. Since $\beta_1 = \alpha_1\mu_1^{-1}$ and $\beta_2 = \alpha_2\mu_2^{-1}$, hence $\alpha_1\mu_1^{-1} = L_{a_1}\varphi$ and $\alpha_2\mu_2^{-1} = R_{a_2}\varphi$, consequently we get (2). The proof of the second part of this theorem eequires only a simple calculation. □

It follows from the proof of the above theorem that the following remark is true.

Remark 2. If a triple of functions $(\alpha_1, \alpha_2, \alpha_3)$ is a solution of equation (1) on groups A and B, then there exists a homomorphism $\varphi : A \to B$ of the groups A and B such that

$$\alpha_1 = L_{a_1}\varphi\mu_1, \quad \alpha_2 = R_{a_2}\varphi\mu_2, \quad \alpha_3 = L_{a_1}R_{a_2}\varphi,$$

where $a_1 = (\alpha_1\mu_1^{-1})(1)$ and $a_2 = (\alpha_2\mu_2^{-1})(1)$.

Remark 3. If $\mu_1 = \mu_2 = id_A$, then equation (1) is said to be the Pexider equation on groups A nad B.

Taking into account the above results we shall give the general solution of the Pexider equation on 3-adic groups.

Theorem 4. *Let $A()$ and $B[\,]$ be 3− adic groups with retracts (A, α, a) and (B, β, b), respectively.*

If a sequence of functions $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is a solution of the Pexider equation on the 3− adic groups $A()$ and $B[\ ]$, then there exist a homomorphism $\varphi : A \to B$ of the groups (A, α, a) and (B, β, b), and elements $a_1, a_2, a_3 \in B$ such that

$$\begin{aligned} \alpha_1 &= L_{a_1}\varphi, \\ \alpha_2 &= \beta^{-1}R_{a_2}\varphi\alpha, \\ \alpha_3 &= L_{a_2b}^{-1}R_{a_3}\varphi L_a, \\ \alpha_4 &= L_{a_1}R_{a_3}\varphi. \end{aligned} \tag{3}$$

If $\varphi : A \to B$ is a homomorphism of the groups (A, α, a) and (B, β, b), and $a_1, a_2, a_3 \in B$ are arbitrary elements, then a sequence of functions

$(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ of form (3) is a solution of the Pexider equation on 3− adic groups $A()$ and $B[\,]$.

Proof. (i) Let a sequence of functions $(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ be a solution of the Pexider equation on the 3− adic groups $A()$ and $B[\,]$, that is $\alpha_4((x_1,x_2,x_3)) = [\alpha_1(x_1),\alpha_2(x_2),\alpha_3(x_3)]$ for all $x_1,x_2,x_3 \in A$.

Using the retracts we can rewrite this equation in the following form

$$\alpha_4(x_1\alpha(x_2)ax_3) = \alpha_1(x_1)(\beta\alpha_2)(x_2)b\alpha_3(x_3)$$

for all $(x_1\alpha(x_2)ax_3) \in A$. Put $a_1 = \alpha_1(1)$, $a_2 = (\beta\alpha_2)(1), a'_3 = \alpha_3(a^{-1})$. Take $x_3 = a^{-1}$ then $\alpha_4(x_1\alpha(x_2)) = \alpha_1(x_1)(\beta\alpha_2)(x_2)ba'_3$ for all $x_1,x_2 \in A$. Let us put $\gamma_2 = R_{ba'_3}\beta\alpha_2$ and so $\alpha_4(x_1\alpha(x_2)) = \alpha_1(x_1)\gamma_2(x_2)$ for all $x_1,x_2 \in A$.

Notice that $(\gamma_2\alpha^{-1})(1) = \gamma_2(1) = (\beta\alpha_2)(1)ba'_3 = a_2ba'_3$. It follows from Theorem 3 and Remark 2 that there exists a homomorphism $\varphi : A \to B$ of the groups (A,α,a) and (B,β,b) such that

$$\alpha_1 = L_{a_1}\varphi,\ \ \gamma_2 = R_{a_2ba'_3}\varphi\alpha,\ \ \alpha_4 = L_{a_1}R_{a_2ba'_3}\varphi.$$

Hence

$$R_{ba'_3}\beta\alpha_2 = R_{a_2ba'_3}\varphi\alpha,\ \ \beta\alpha_2 = R_{a_2}\varphi\alpha,\ \ \alpha_2 = \beta^{-1}R_{a_2}\varphi\alpha;\quad \text{Thus}$$

$$\alpha_1 = L_{a_1}\varphi,\ \ \alpha_2 = \beta^{-1}R_{a_2}\varphi\alpha,\ \ \alpha_4 = L_{a_1}R_{a_2ba'_3}\varphi.$$

Put $x_1 = 1$ then $\alpha_4(\alpha(x_2)ax_3) = a_1(\beta\alpha_2)(x_2)b\alpha_3(x_3)$ for all $x_2,x_3 \in A$. Let us set $\kappa_2 = L_{a_1}\beta\alpha_2, \kappa_3 = L_b\alpha_3$. Then $\alpha_4(\alpha(x_2)L_a(x_3)) = \kappa_2(x_2)\kappa_3(x_3)$ for all $x_3,x_3 \in A$. Notice that $(\kappa_2\alpha^{-1})(1) = \kappa_2(1) = a_1(\beta\alpha_2)(1) = a_1a_2$ and $(\kappa_3L_a^{-1})(1) = (L_b\alpha_3L_a^{-1})(1) = (L_b\alpha_3)(a^{-1}) = ba'_3$. It follows from Theorem 3 and Remark 2 that there exists a homomorphism $\psi : A \to B$ of the groups (A,α,a) and (B,β,b) such that $\kappa_2 = L_{a_1a_2}\psi\alpha$, $\kappa_3 = R_{ba'_3}\psi L_a$, $\alpha_4 = L_{a_1a_2}R_{ba'_3}\psi$. Since $\alpha_4 = L_{a_1}R_{a_2ba'_3}\varphi$ and so $L_{a_1a_2}R_{ba'_3}\psi = L_{a_1}R_{a_2ba'_3}\varphi$, consequently $\psi = L_{a_2}^{-1}R_{a_2}\varphi$. Moreover, $\kappa_3 = L_b\alpha_3 = R_{ba'_3}\psi L_a$, hence $\alpha_3 = L_b^{-1}R_{ba'_3}L_{a_2}^{-1}R_{a_2}\varphi L_a$, consequently $\alpha_3 = L_{a_2b}^{-1}R_{a_2ba'_3}\varphi L_a$. Putting $a_3 = a_2ba'_3$ we get the sequence $(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ of form (3).

(ii) Suppose that the functions of the sequence $(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ are of form (3).

Then

$$\begin{aligned}\alpha_4((x_1,x_2,x_3)) &= \alpha_4(x_1\alpha(x_2)ax_3)\\ &= a_1\varphi(x_1)(\varphi\alpha)(x_2)\varphi(a)\varphi(x_3)a_3\\ &= (a_1\varphi(x_1))((\varphi\alpha)(x_2)a_2)(a_2^{-1}\varphi(a)\varphi(x_3)a_3)\\ &= (a_1\varphi(x_1))((\varphi\alpha)(x_2)a_2)b(b^{-1}a_2^{-1}\varphi(ax_3)a_3)\\ &= ((L_{a_1}\varphi)(x_1))((R_{a_2}\varphi\alpha)(x_2))b((L_{a_2b}^{-1}R_{a_3}\varphi L_a)(x_3))\\ &= \alpha_1(x_1)(\beta\alpha_2)(x_2)b\alpha_3(x_3) = [\alpha_1(x_1),\alpha_2(x_2),\alpha_3(x_3)]\end{aligned}$$

for all $x_1, x_2, x_3 \in A$.
This completes the proof of the theorem.

It follows from the proof of the above theorem that the following remark is true.

Remark 4. Let $A()$ and $B[\,]$ be 3− adic groups with retracts (A,α,a) and (B,β,b), respectively.

If a sequence of functions $(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ is a solution of the Pexider equation on the 3− adic groups $A()$ and $B[\,]$, then there exists a homomorphism $\varphi : A \to B$ of the groups (A,α,a) and (B,β,b) such that the functions $(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ are of form (3) for $a_1 = \alpha_1(1)$, $a_2 = (\beta\alpha_2)(1)$, $a_3 = a_2b\alpha_3(a^{-1})$.

The Cauchy equation on 3− adic groups $A()$ and $B[\,]$ is the following functional equation

$$f((x_1,x_2,x_3)) = [f(x_1), f(x_2), f(x_3)] \tag{4}$$

for arbitrary $x_1, x_2, x_3 \in A$, where $f : A \to B$ is an unknown function.

Theorem 5. *Let $A()$ and $B[]$ be 3− adic groups with retracts (A,α,a) and (B,β,b), respectively. A function $f : A \to B$ is a solution of the Cauchy equation on the 3− adic groups $A()$ and $B[\,]$ if and only if there exists a homomorphism $\varphi : A \to B$ of the groups (A,α,a) and (B,β,b), and an element $a_1 \in B$ such that*

(a) $f(x) = a_1\varphi(x)$,

(b) $(\varphi\alpha)(x)\beta(a_1) = \beta(a_1)(\beta\varphi)(x)$,

(c) $\varphi(a) = \beta(a_1)ba_1$

for every $x \in A$.

Proof. Let a function $f : A \to B$ be a solution of equation (4). It follows from Theorem 4 and Remark 4 that there exists a homomorphism $\varphi : A \to B$ of the groups (A, α, a) and (B, β, b) such that $f(x) = a_1\varphi(x)$, $(\varphi\alpha)(x)\beta(a_1) = (\beta f)(x)$ for $a_1 = f(1)$ and for every $x \in A$. Hence $(\varphi\alpha)(x)\beta(a_1) = \beta(a_1\varphi(x))$, $(\varphi\alpha)(x)\beta(a_1) = \beta(a_1)(\beta\varphi)(x)$ for every $x \in A$. Moreover, taking $x_1 = x_2 = x_3 = 1$ in equation (4) and applying the retracts we get $f(a) = a_1\beta(a_1)ba_1$ and so $\varphi(a) = \beta(a_1)ba_1$.

Thus we have obtained conditions $(a), (b), (c)$.

Let us suppose that there exist a homomorphism $\varphi : A \to B$ of the groups (A, α, a) and (B, β, b), and an element $a_1 \in B$ such that conditions $(a), (b), (c)$ are fulfilled. It is easy to verify that the function $f(x) = a_1\varphi(x)$ satisfies equation (4) for every $x \in A$.

This completes the proof of the theorem. □

Remark 5. Theorem 5 is a particular case of Theorem 1 (given in Corovei [1]).

Remark 6. If 3– adic groups $A()$ and $B[\,]$ are commutative, then condition (b) of Theorem 5 is always trivially fulfilled.

We shall give an example of two 3- adic groups for which the Cauchy equation does not have any solution.

Example 3. Let $A()$ be the 3- adic group occurring in Example 1. Notice that (A, α, e) is a retract for the 3- adic group $A()$ (with the operation in the Klein group).

Consider the group $B = \{1, 2, 3, 4\}$ endowed with the operation:

	1	2	3	4
1	1	2	3	4
2	2	3	4	1
3	3	4	1	2
4	4	1	2	3

The 3-ary operation on the set B is defined as follows

$$[x_1, x_2, x_3] = 2x_1x_2x_3$$

for all $x_1, x_2, x_3 \in B$.

$B[\ ]$ forms a 3-adic group for which $(B, id_B, 2)$ is a retract (with the operation defined by means of the above table).

Suppose that a function $f : A \to B$ is a solution of the Chauchy equation on the 3-adic groups $A()$ and $B[\ ]$. According to Theorem 5 there exists a homomorphism $\varphi : A \to B$ of the groups (A, α, e) and $(B, id_B, 2)$, and an element $a_1 \in B$ such that $\varphi(e) = a_1 2 a_1$. Since $\varphi(e) = 1$, it is eassy to check that this equality is not fulfilled for any element $a_1 \in B$.

We shall give the definitions of the conditional Cauchy equation and the redundant condition on 3-adic group. These definitions are analogues of the suitable definitions for groups (cf. [2]).

Let $A()$ and $B[\ ]$ be 3-adic groups. Assume that $Z \subset A^3$ and $Z \neq \emptyset$. We say that a function $f : A \to B$ is a solution on the 3-adic groups $A()$ and $B[\ ]$ of the conditional Cauchy equation relative to Z if

$$f((x_1, x_2, x_3)) = [f(x_1), f(x_2), f(x_3)]$$

for all (x_1, x_2, x_3) in Z.

If an arbitrary solution: $f : A \to B$ on the 3-adic groups $A()$ and $B[\ ]$ of the conditional Cauchy equation relative to Z is a solution of the Cauchy equation on the whole set A^3, then we say that the condition (Z, A, B) is redundant.

Let $A()$ and $B[\ ]$ be 3-adic groups. Let A_0 be a 3-adic subgroup of the 3-adic group $A()$. For the 3-adic group $A()$ we construct the retract (A, α, a) by means of the Sokolov method taking

$$xy = (x, a_0, y)$$

for all $x, y \in A$ and for an arbitrary fixed element $a_0 \in A_0$. Then A_0 forms a subgroup of the group (A, α, a) and $(A_0, \alpha_{|A_0}, a)$ is a retract for the 3-adic group $A_0()$. Let (B, β, b) be an arbitrary fixed retract of the 3-adic group $B[\ ]$. Suppose that $Z = A_0^3$.

Applying the above assumptions, notations, and Theorem 5 we get

Theorem 6. *A function $f : A \to B$ is a solution on 3-adic groups $A()$ and $B[\ ]$ of the conditional Cauchy equation relative to Z if and only if there exists a homomorphism $\varphi : A_0 \to B$ of the groups $(A_0, \alpha\,|_{A_0}, a)$ and (B, β, b), and element $a_1 \in B$ such that*

(a) $f(x) = a_1\varphi(x)$,

(b) $(\varphi\alpha)(x)\beta(a_1) = \beta(a_1)(\beta\varphi)(x)$,

(c) $\varphi(a) = \beta(a_1)ba_1$

for every $x \in A_0$.

In virtue of Theorem 2 we get

Corollary 2. *Let* $A()$ *and* $B[\ ]$ *be 3-adic groups. Suppose that* $Z = A_1 \times A \times A$ *or* $Z = A \times A \times A_3$ *for non-empty subsets* A_1 *and* A_3 *of the set* A. *Then the condition* (Z, A, B) *is redundant.*

Let $A()$ and $B[\ ]$ be 3-adic groups. Assume that $A_2 \subset A$ and $A_2 \neq \emptyset$. Consider the set $Z = A \times A_2 \times A$.

The condition (Z, A, B) may not be redundant (cf. Example 1). Without loss of generality we may assume that the identity of a certain retract (A, α, a) of the 3-adic group $A()$ is an element of the set A_2. Indeed, notice that $x = (x, \bar{x}, \bar{\bar{x}}) = \bar{\bar{\bar{x}}}$ for every $x \in A$. Let (B, β, b) be an arbitrary fixed retract of the 3-adic group $B[\ ]$.

Applying the above assumptions and notations we obtain

Theorem 7. *A function* $f : A \to B$ *is a solution on 3-adic groups* $A()$ *and* $B[\ ]$ *of the conditional Cauchy equation relative to* Z *if and only if there exists a homomorphism* $\varphi : A \to B$ *of the groups* (A, α, a) *and* (B, β, b), *and an element* $a_1 \in B$ *such that*

(a) $f(x) = a_1\varphi(x)$ *for every* $x \in A$,

(b) $(\varphi\alpha)(x)\beta(a_1) = \beta(a_1)(\beta\varphi)(x)$ *for every* $x \in A_2$,

(c) $\varphi(a) = \beta(a_1)ba_1$.

Proof. (i) Let a function $f : A \to B$ be a solution on the 3-adic groups $A()$ and $B[\ ]$ of the conditional Cauchy equation relative to Z. Then $f((x_1, x_2, x_3)) = [f(x_1), f(x_2), f(x_3)]$ for all $x_1, x_3 \in A$ and $x_2 \in A_2$. Hence $f(x_1\alpha(x_2)ax_3) = f(x_1)(\beta f)(x_2)bf(x_3)$ for all $x_1, x_3 \in A$ and $x_2 \in A_2$. Let us put $f(1) = a_1$. If $x_2 = 1$ then $f(x_1ax_3) = f(x_1)\beta(a_1)bf(x_3)$, and consequently $f(x_1L_a(x_3)) = f(x_1)(L_{\beta(a_1)b}f)(x_3)$ for all $x_1, x_3 \in A$. In virtue of Theorem 3 and Remark

2 we obtain that $f = L_{a_1}\varphi$ for a certain homomorphism $\varphi : A \to B$ of the groups (A, α, a) and (B, β, b). Consequently condition (a) holds.

Taking $x_1 = x_2 = x_3 = 1$ we have $f(a) = a_1\beta(a_1)ba_1$, hence $\varphi(a) = \beta(a_1)ba_1$ and condition (c) is fulfilled.

Notice that $f(a^{-1}) = b^{-1}\beta(a_1)^{-1}$. Indeed, $\varphi(a^{-1}) = \varphi(a)^{-1} = a_1^{-1}b^{-1}\beta(a_1)^{-1}$. Hence $f(a^{-1}) = a_1\varphi(1^{-1}) = b^{-1}\beta(a_1)^{-1}$. Put $x_1 = 1, x_3 = a^{-1}$ and assume that $x_2 \in A_2$. Then

$$f(\alpha(x_2)) = a_1\beta(f(x_2))bf(a^{-1}),$$

$$f(\alpha(x_2)) = a_1\beta(f(x_2))b(b^{-1}\beta(a_1)^{-1}),$$

$$a_1\varphi(\alpha(x_2))\beta(a_1) = a_1\beta(a_1\varphi(x_2)),$$

$$(\varphi\alpha)(x_2)\beta(a_1) = \beta(a_1)(\beta\varphi)(x_2).$$

Thus condition (b) holds.

(ii) We assume that there exists a homomorphism $\varphi : A \to B$ of the groups (A, α, a) and (B, β, b), and an element $a_1 \in B$ such that conditions $(a), (b), (c)$ are fulfilled. Then for all $x_1, x_3 \in a$ and $x_2 \in A_2$ we have

$$f((x_1, x_2, x_3)) = f(x_1\alpha(x_2)ax_3)$$

$$= a_1\varphi(x_1\alpha(x_2)ax_3) = a_1\varphi(x_1)(\varphi\alpha)(x_2)\varphi(a)\varphi(x_3)$$

$$= a_1\varphi(x_1)(\varphi\alpha)(x_2)\beta(a_1)ba_1\varphi(x_3) = (a_1\varphi(x_1))(\beta(a_1)(\beta\varphi)(x_2)b(a_1\varphi(x_3))$$

$$= (a_1\varphi(x_1))\beta(a_1\varphi(x_2))b(a_1\varphi(x_3)) = f(x_1)(\beta f)(x_2)bf(x_3)$$

$$= [f(x_1), f(x_2), f(x_3)].$$

This completes the proof of the theorem. □

Remark 7. If 3-adic groups $A()$ and $B[\,]$ are commutative, then the condition $(A \times A_2 \times A, A, B)$ for $\emptyset \neq A_2 \subset A$ is redundant (cf. Remark 1).

References

[1] Corovei I., Purdea I., Relations between the homomorphisms of n-groups and the homomorphisms of their reduces, Mathematica (Cluj), 25 (1983), 131-135.

[2] Dhombres J., Some aspects of functional equations, Chulalongkorn University Press, Bangkok, 1979.

[3] Dudek W., Michalski J., On generalization of Hosszú theorem, Demonstratio Math., 15 (1982), 783-805.

[4] Hosszú M., On the explicit form of n-group operationss, Publ. Math., Debrecen, 10 (1963), 88-92.

[5] Kuczma M., An introduction to the theory of functional equations and inequalities, Państwowe Wydawnictwo Naukowe, Warszawa - Kraków - Katowicw, 1985.

[6] Monk J.D., Sioson F.D., On general theory of m-groups, Fund. Math., 72 (1971), 233-244.

[7] Sokolov E.I., The Gluskin - Hosszú theorem for the Dörnte n - groups (Russian), Nets and quasigroups, Mat. Issled., 39 (1976), 187-189.

REZIME

O USLOVNOJ KOŠIJEVOJ JEDNAČINI NA TERNARNIM GRUPAMA

U radu su razmotrene dve vrste Košijeve jednačine na ternarnim grupama i na osnovu dobijenih rezultata dato je opšte rešenje Peksiderove jednačine na tim grupama. Takodje su odredjene i neke karakterizacije Košijevog i - jezgra za funkcije definisane na n - arnim grupama.

Received by the editors January 13, 1988