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THE LIMITING SPECTRUM OF INFINITE GRAPHS

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Abstract

In this paper, in a natutal way, the limiting spectrum of an infinite countable connected graph is defined. This spectrum is real, discrete and a graph invariant. By this definition several properties of the spectrum of infinite graphs to the infinite case are generalized. Besides, some new properties and questions arise.

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1. Basis results

Let $G = (V, E)$ be an infinite countable connected graph without loops or multiple edges. In the sequel, we will simply call it a graph.

So far, the spectrum of infinite graphs has been investigated in literature in at least two ways. We mention the approach by A. Torgašev ([5],[6], [7],[8] etc) and the approach by B. Mohar ([3],[4] etc). Both approaches are described in several details in book [2].

The advantage of the approach by A. Torgašev is that the spectrum is real and discrete. The disadvantage lies in the fact that it uses a weighted adjacency matrix, depending on a parameter a $(0 < a < 1)$. Consequently, the spectrum obtained in this way is not a graph invariant.

The approach by B. Mohar has the advantage that the spectrum obtained is real and is a graph invariant. But it is not discrete in the general case.

In the present paper we will introduce, in a very simple and natural way, a new definition of the spectrum of an infinite graph. The obtained limiting spectrum is real, discrete and a graph invariant. The only unpleasant feature of this kind of spectrum is that some of its values can be equal to $\pm\infty$. But, as we have already said, we are dealing with infinite graphs.

If G is an infinite graph, let $\mathcal{F}$ be the set of all finite connested induced subgraphs of G.

For any positive integer n, the n^{th} positive limiting eigenvalue $\lambda_n^+(G)$ of G (in short, LEV) is defined by

$$\lambda_n^+(G) = \sup\{\lambda_n^+(F) \mid F \in \mathcal{F}\} \leq +\infty,$$

if the n^{th} positive eigenvalue $\lambda_n^+(F)$ exists for at least one graph $F \in \mathcal{F}$.

Similarly, the n^{th} negative limiting eigenvalue $\lambda_n^-(G)$ of G is defined by

$$\lambda_n^-(G) = \inf\{\lambda_n^-(F) \mid F \in \mathcal{F}\} \geq -\infty,$$

if the n^{th} negative eigenvalue $\lambda_n^-(F)$ exists for at least one graph $F \in \mathcal{F}$.

We note that $\lambda_n^+(G) = r(G)$ (the spectral radius or the largest limiting eigenvalue of G) and $\lambda_n^-(G) = \lambda(G)$ (the least limiting eigenvalue of G) always exist as finite or infinite numbers. The n^{th} positive LEV $\lambda_n^+(G)$ or the n^{th} negative LEV $\lambda_n^-(G)$ cannot exist for some sufficiently large n (and then for all $m \geq n$). For instance, if G is the complete infinte graph K_∞, then each finite induced subgraph of G is also complete, and we easily find that $r(G) = +\infty$, all the other positive LEV's $\lambda_n^+(G)$ $(n \geq 2)$ do not exist, and

$$\lambda(G) = \lambda_2^-(G) = \lambda_3^-(G) = \ldots = -1.$$

For any infinite graph G, the limiting spectrum $\sigma_L(G)$ of G is defined to be the sequence of all its positive and all its negative LEV's:

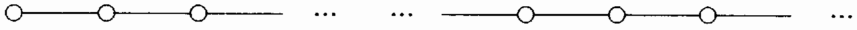
$$\lambda_1^+(G) \geq \lambda_2^+(G) \geq \cdots > 0 > \cdots \lambda_2^-(G) \geq \lambda_1^-(G). \tag{1}$$

By definition, $\sigma_L(G)$ consists of only nonzero values, including also their "multiplicities". This spectrum is obviously real and discrete in any case.

Since $|\lambda(G)| \leq r(G) \leq +\infty$, the whole limiting spectrum $\sigma_L(G)$ is situated in the (finite or infinite) interval $[-r(G), r(G)]$.

Note that both extremal cases, when $\lambda(G)$, $r(G)$ are finite, or $\lambda(G) = -\infty$, $r(G) = +\infty$ are possible.

For instance, if G is the one-side infinite path P_∞^+, or the two-side infinite path $P_\infty^\pm$, then $\lambda(G) = -2$ and $r(G) = +2$.



P_∞^+ $P_\infty^\pm$

Moreover, then $\sigma_L(g) = \{2, 2, \ldots; -2, -2, \ldots\}$.
The mentioned graphs provide an example for two nonisomorphic cospectral infinite graphs. They also show that the spectral radius can not be simple limiting eigenvalue.

On the other hand, if G is the complete bipartite graph $K_{\infty,\infty}$, then it is easy to see that $\lambda(G) = -2$, $r(G) = +2$, and $\sigma_L(G) = \{+\infty, -\infty\}$.

In the general case, some limiting eigenvalues of an infinite graph are infinite, and all others are finite, that is, we have a mixed case.

On of the most important properties of the spectrum defined in this way is that it is a graph invariant, that is it does not depend on the way of labelling of its vertex set $V(G)$. This can be easily proved by the known interlacing theorem for finite graphs [1, p.19].

We also note the values $r(G)$ and $\lambda(G)$ have already been treated in literature. So, the invariant $r(G)$ has been treated in [3] by B. Mohar, in his approach to the spectrum of infinite graphs. Paper [8] describes all the infinite graphs with the property $r(G) \leq \sqrt{2+\sqrt{5}}$. Papers [9] and [11] refer to infinite graphs with the property $\lambda(G) \geq -2$. In particular, in paper [3] all the infinite graphs with the property $r(G) < +\infty$ have been determined.

Theorem 1. *[3] An infinite graph G has a finite spectral radius $r(G)$ if and only if it has uniformly bounded vertex degrees.*

If the spectral radius of a graph G is finite, then obviously, all its limiting eigenvalues are also finite. An interesting property of such graphs is proved in the following theorem.

Theorem 2. *Let the spectral radius of an infinite graph G be finite. Then* *its limiting specrtum $\sigma_L(G)$ consists of infinitely many positive and infinitely many negative LEV's.*

In this case we also have

$$|\lambda| \geq 2, \quad \textit{for every value } \lambda \in \sigma_L(G). \tag{2}$$

Relation (2) means that the entire spectrum $\sigma_L(G)$ lies in the set $[\lambda(G), -2] \cup [2, r(G)]$.

Proof. By Theorem 1, graph G will have uniformly bounded vertex degrees. Since it is connected and infinite, it is easy to see that there is a sequence of paths P_n $(n = 2, 3, \ldots)$, which are induced subgraphs of G. We denote this fact by $P_n \subseteq G$ $(n \leq 2)$.

Next, let m be an arbitrary positive integer. Since all the graphs $P_n \subseteq G$ $(n \geq 2m)$ possess the m^{th} positive and m^{th} negative eigenvalue, we have a similar property for graph G. Hence, G will have infinitely many positive and infinitely many negative limiting eigenvalues.
Besides, since

$$\lambda_m^+(P_n) \to 2, \quad \lambda_m^-(P_n) \to -2, \quad \text{as } n \to \infty,$$

we get $\lambda_m^+(G)$, $|\lambda_m^-(G)| \geq 2$ for every index $m \geq 1$.

This completes the proof. □

Remark. The statement $\sigma_L(G) \cap (-2, 2) = \emptyset$ can be false, if $r(G) = +\infty$. For instance, if G is the complete 3-partite graph $K_{1,1,\infty}$, then $\sigma_L(G) = \{+\infty; -1, -\infty\}$.

By the corresponding properties of finite graphs, we also find the following results.

Proposition 1.

- *(a) The limiting spectrum of an arbitrary bipartite infinite graph is symmetric around zero.*

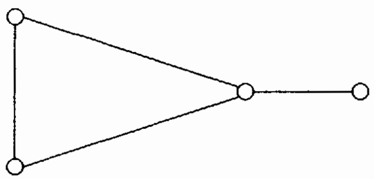
- *(b) An infinite graph G has exactly one positive limiting eigenvalue if and only if it is a complete m-partite graph ($m \leq +\infty$). In this case $r(G) = +\infty$.*

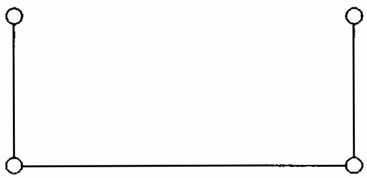
We do not know if the converse statement of Proposition 1 (a) is true. That is, we do not know if there exists a nonbipartite infinite graph whose limiting spectrum (together with the corresponding multiciplities) is symmetric around zero.

We also do not know the necessary and sufficient conditions under which an infinite graph has a finite least limiting eigenvalue $\lambda(G)$). The example of graph K_∞ shows that $r(G)$ can be equal to $+\infty$, while $\lambda(G)$ is finite. But if $\lambda(G) = -\infty$ then we obviously have that $r(G) = +\infty$.

There are also many unsolved question concerning the n^{th} positive and the n^{th} negative limiting eigenvalue of infinite graphs ($n \geq 2$).

If $n = 2$, by the corresponding results for finite graphs ([2]), we know that $\lambda_2^+(G)$ exists if and only if G has one of the following graphs





as an induced subgraph. Also, $\lambda_2^-(G)$ exists if and only if G has one of graphs K_3, P_4 as an induced subgraph. We also know similar criterions for the existence of the limiting eigenvalues $\lambda_3^-(G)$ and $\lambda_4^-(G)$, but not for any other limiting eigenvalue of infinite graphs.

Besides, we do not know any necessary and sufficient condition under which $\lambda_n^+(G)$ (or $\lambda_n^-(G)$), for a fixed $n \geq 2$, is finite.

2. Infinite graphs with finitely many LEV's

In this section we will describe all the infinite graphs which have finitely many limiting eigenvalues.

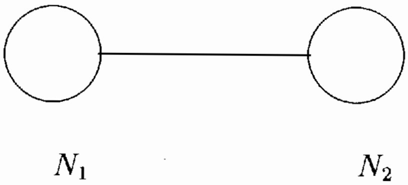
First, we consider the following equivalence relation α on the vertex set $V(G)$ of an infinite graph G. Two vertices x and y of G are in relatin α if they are nonadjacent in G and they have exactly the same neighbours in G. The corresponding quotient graph g is called the canonical graph of G. It is also connected and we obviously have that $g \subseteq G$. As the following examples show, the graph g can be finite as well as infinite.

If G is any complete m-partite graph ($m < \infty$), then its canonical graph is the complete finite graph K_m. If G is the complete graph K_∞, then its canonical graph is the same graph K_∞.

We call the graph G canonical if it has no two equivalent vertices. We say that G is of a finite type k if the corresponding canonical graph g has $k < \infty$ vertices. In the other case, we say that G is of an infinite type.

Let G be an infinite graph, and let $N_1, N_2, \ldots$ be the corresponding sets of equivalent vertices in G. It is easy to see that each set N_i ($i \in V(g)$) consists of isolated vertices only, and for every two indices $i, j \in V(G)$ ($i \neq j$) the sets N_i, N_j are either completely adjacent or completely nonadjacent in G. This means that if an edge between these sets is present in G, then all other such edges are also present. Hence, we often represent the sets N_i and N_j ($i \neq j$) by only one such edge. So, the complete bipartite graph $G = K_{N_1,N_2}$ can be represented as follows:

$G_1 =$



In the general case, we sometimes also write $G = g(N_1, N_2, \ldots)$ ($i \in V(g)$).

The canonical graphs of infinite graphs have been investigated (by a different approach to the spectrum of infinite graphs) in papers [6], [7] etc.

In the sequel, we will consider infinite graphs with finitely many limiting eigenvalues, and give a full classification of such graphs.

Theorem 3. *An infinite graph G has finitely many limiting eigenvalues if and only if it is of a finite type. In this case $r(G) = +\infty$ and $\lambda(G) = -\infty$.*

Proof. Assume, first, that G has a finite number m of limiting eigenvalues. Then any graph $F \in \mathcal{F}$ has at most m nonzero eigenvalues. By ([6], Theorem 2) the canonical graph f of the graph F will have at most $2^m - 1$ vertices. Whence, it is easy to see that the canonical graph g of G also has at most $2^m - 1$ vertices, that is G is a graph of a finite type.

Conversely, assume that G is of a finite type k, that is $|g| = k < \infty$. Then any graph F has at most k nonzero eigenvalues. Since this holds for every graph $F \in \mathcal{F}$, it is easy to see that G has at most k limiting eigenvalues, Q. E. D.

Next, assume that G is of a finite type k, that is $|g| = k < \infty$. Then at least one of the characteristic subsets $N_1, \ldots, N_k$ of G must be infinite. Therefore, G will possess a sequence of stars $K_{1,n}$ $(n = 1, 2, \ldots)$, which are induced subgraphs of G. By Theorem 1 this gives $r(G) = +\infty$.

Now, assume that $\lambda(G) > -\infty$. Then all the nagative limiting eigenvalues of G must be finite, and the sum af all the negative LEV's must be also finite. But since $\sum \lambda_i(F) = 0$ for any graph $F \in \mathcal{F}$, and $r(G) = \sup\{r(F)|\ F \in \mathcal{F}\} = +\infty$, we easily get a contrdiction. Hence, we have $\lambda(G) = -\infty$, which completes the proof. □

The infinite graph $K_{1,\infty}$ provides a typical example for graphs mentioned in Theorem 3. In this case we have $\sigma_L(G) = \{+\infty; -\infty\}$.

In the general case, the limiting eigenvalues of graphs mentioned in Theorem 3, other than $r(G)$ and $\lambda(G)$, can be finite as well as infinite.

Infinite graphs of the finite type with all LEV's equal to $\pm\infty$ attract special attention. We note a particular class of infinite graphs with the mentioned property.

Proposition 2. *Let G be a finite type infinite graph with all the characteristic subsets infinite. Then all its limiting eigenvalues equal to $\pm\infty$.*

Proof. Let g be the canonical graph of the graph G; then $|g| = k < \infty$. Let $\lambda_1, \ldots, \lambda_r$ be all the nonzero eigenvalues of the graph g (together with their multiciplities). Note that $g \in \mathcal{F}$. For an arbitrary positive integer n, let F_n be the (connected) graph induced by choosing n arbitrary vertices in each of the subsets N_i of the graph G ($i = 1, \ldots, k$). It is not difficult to see that all the nonzero eigenvalues of the graph F_n are of te form $n\lambda_1, \ldots, n\lambda_r$ and that $F_n \in \mathcal{F}$ $(n = 1, 2, \ldots)$. Since $n\lambda_i \to +\infty$ (if $\lambda_i > 0$) and $n\lambda_i \to -\infty$ (if $\lambda_i > 0$) as $n \to \infty$, we immediately get the statement. □

It would also be interesting to give a full classification of infinite graphs of the finite type, all of whose LEV's equal $\pm\infty$. But, so far, we have no such classification.

As the graph $K_{1,\infty}$ shows, the condition which appears in the last proposition is not necessary in the general case.

3. On infinite graphs with all LEV's equal to $\pm\infty$

In this section we examine whether there exists an infinite graph whose limiting spectrum consists of m times $+\infty$ and n times $-\infty$ $(m, n \leq +\infty)$.

For any two positive integers $m, n \leq +\infty$, let $Q(m, n)$ be the set of all nonisomorphic infinite graphs whose limiting spectrum consists of m times $+\infty$ and n times $-\infty$. It is obvious that any graph $G \in Q(m, n)$ $(m, n \leq +\infty)$ does not have uniformly bounded vertex degrees.

Proposition 3.

(a) The class $Q(m, \infty)$ is nonempty for every positive integer $m \leq +\infty$. In particular, the class $Q(\infty, \infty)$ is nonempty.

(b) The class $Q(m, n)$ $(n < \infty)$ is nonempty for $m = 1, 2, \ldots, f(n)$, where $f(n)$ is a certain finite function of $n \in \mathbf{N}$, and empty for all other $m > f(n)$. In particular, the class $Q(\infty, n)$ is empty for any positive integer n.

Proof. For any positive integer $m \leq +\infty$, let h_m be the graph obtained by identification of a point of the graph K_∞ with an end-point of the path P_{2m-1} (that is, with the only end-point of the graph P_∞^+ if $m = \infty$).

By the corresponding results of the paper [10], we find that graph h_m has exactly m positive eigenvalues and ∞ negative eigenvalues.

Next, let H_m be the infinite graph whose canonical graph is h_m, and all of whose characteristic subsets are infinite. We immediately find that $H_m \in Q(m, \infty)$.

The remaining part of this statement can be proved in a similar manner, by corresponding results on the numbers of positive and negative eigenvalues of finite graphs ([12]). □

It would also be an interesting (and hard enough) question to describe all the infinite graphs from the nonempty classes $Q(m, n)$ $(m, n \leq +\infty)$.

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REZIME

OGRANIČAVAJUĆI SPEKTAR BESKONAČNIH GRAFOVA

U ovom radu, na prirodan način se definiše ograničavajući spektar beskonačno povezanog grafa. Ovaj spektar je realan, diskretan i grafovska invarijanta. Ovom definicijom, nekoliko osobina spektra konačnih grafova uopštene su za beskonačni slučaj. Pored toga, pojavljuju se neka nova svojstva i nameću neka nova pitanja.

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