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ON A CLASS OF SECOND FUNCTIONAL EQUATIONS OF ALTERNATIVE FUNCTIONS

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Abstract

In this paper the necessary and sufficient conditions for the existence of solutions of a class of second order functional equations of alternative functions are considered. These solutions are also given in their explicit form.

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Let $(P_2, +, \cdot)$ be given, where $+$ and $\cdot$ are binary operations of addition and multiplication (mod 2), $L_2 = \{0, 1\}$ and $(a \cdot b = ab)$.

A function $f : L_2^n \longrightarrow L_2$ is called an alternative function; L_2^n is the direct power of L_2.

Definition 1.

(a) *Partial derivatives of an alternative function $f : L_2^n \longrightarrow L_2$ at the variables $x_i (i = 1, 2, \ldots n)$ are functions*

$$\frac{\partial f_\alpha}{\partial x_i} : L_2^n \longrightarrow L_2 \text{ defined by}$$

$$\frac{\partial f_\alpha}{\partial x_i}(X) = f(x_1, \ldots, x_{i-1}, \alpha, x_{i+1}, \ldots, x_n) + f(X)$$

$$\alpha \in L_2, \quad 1 \leq i \leq n, \quad \text{where} \quad X = (x_1, \ldots, x_n)$$

(b) Partial derivatives of a higher order are functions

$$\frac{\partial^m f\alpha_{i_1}\dots\alpha_{i_m}}{\partial x_{i_1}\dots\partial x_{i_m}} =$$

$$= \frac{\partial}{\partial x_{i_m}}(\dots(\frac{\partial}{\partial x_{i_2}}(\frac{\partial f\alpha_{i_1}}{\partial x_{i_1}})\alpha_{i_2})\dots)\,\alpha_{i_m}$$

It is obvious from Definition 1 that for every $\alpha, \beta \in L_2$ and for every couple of alternative functions f and g the following properties hold:

$$(1)\qquad \begin{aligned} \frac{\partial c_\alpha}{\partial x_i} &= 0, \ \frac{\partial (cf)_\alpha}{\partial x_i} = c\,\frac{\partial f_\alpha}{\partial x_i}, \\ \frac{\partial (f+g)\alpha}{\partial x_i} &= \frac{\partial f_\alpha}{\partial x_i} + \frac{\partial g_\alpha}{\partial x_i}, \\ \frac{\partial (f\cdot g)_\alpha}{\partial x_i} &= \frac{\partial f_\alpha}{\partial x_i}\,g + f\,\frac{\partial g_\alpha}{\partial x_i} + \frac{\partial f_\alpha}{\partial x_i}\cdot\frac{\partial g_\alpha}{\partial x_i} \\ \frac{\partial^2 f_{\alpha\beta}}{\partial x_i \partial x_j} &= \frac{\partial^2 f_{\beta\alpha}}{\partial x_j \partial x_i}, \ i \neq j \\ \frac{\partial^m f_{\alpha\alpha\cdots\alpha}}{\partial x_i^m} &= \frac{\partial f_\alpha}{\partial x_i}, \ m > 1, \ \underbrace{\alpha\alpha\dots\alpha}_{m\ times} \end{aligned}$$

Lemma 1. *A functional equation with an unknown alternative function f*

$$(2)\qquad \frac{\partial f_{\alpha i}}{\partial x_i} = g(X) \quad , \ where \quad \alpha_i \in L_2$$

has a solution if and only if $g(x_1,\dots,x_{i-1},\alpha,x_{i+1},\dots,x_n) = 0.$ *All functions f that are solutions are determined by the formula*

$$(3)\qquad f(X) = c(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n) + g(X),$$

where c is an arbitrary function with the variables

$$x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n$$

Proof. First, let us introduce the following abbreviations

$$(\tilde{x}_i) = (x_1,\dots,x_{i-1},x_{i+1},\dots,x_n), \ (\tilde{\tilde{\alpha}}_i) = (x_1,\dots,x_{i-1},\alpha_i,x_{i+1},\dots,x_n).$$

Substituing x_i with α_i in equation (2) we get

$$f(\tilde{\tilde{\alpha}}_i) + f(\tilde{\tilde{\alpha}}_i) = g(\tilde{\tilde{\alpha}}_i), \text{ the condition } g(\tilde{\tilde{\alpha}}_i) = 0$$

Conversely, let us suppose that the given condition is satisfied and let us determine all functions f.

$$f(\tilde{\tilde{\alpha}}_i) + f(X) = C(\tilde{x}_i) + g(\tilde{\tilde{\alpha}}_i) + (c(\tilde{x}_i) + g(X)) = g(X),$$

because $g(\tilde{\alpha}_i) = 0$. Hence, it remains to prove that every solution f is of the form (3).

Let f be *a* solution and let us find the appropriate form of equation (2); we can conclude that $f(X) = f(\tilde{\alpha}_i) + f(X)$. $f(\tilde{\tilde{\alpha}}_i)$ is *a* function only of $x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots x_n)$ (it does not depend on x_i), therefore $f(X)$ is of the form (3).

Lemma 2. *A system of functional equations of alternative functions*

$$\begin{aligned} \frac{\partial f_\alpha}{\partial x} &= r_1 x + r_2 y + r_3 xy + r, \\ \frac{\partial f_\beta}{\partial y} &= g_1 x + g_2 y + g_3 xy + g, (x,y) \in L_2' \end{aligned}$$

(where $r_1, r_2, r_3, r, g_1, g_2, g_3, g$ are constants from L_2), has a solution if and only if the following conditions are satisfied

$$(4.1) \quad \begin{aligned} r_3\alpha + r_2 &= 0 \\ r_1\alpha + r &= 0 \\ g_2\beta + g_1 &= 0 \\ g_2\beta + g &= 0 \end{aligned} \qquad (4.2) \quad \begin{aligned} r_3 + g_3 &= 0 \\ r_2 + g_3\alpha &= 0 \\ r_3\beta + g_1 &= 0 \\ r_2\beta + g_1\alpha &= 0 \end{aligned}$$

All the solutions are determined by the formula

$$(4.3) \qquad f(x,y) = c + r_1 x + g_1 x + g_2 y + g_3 xy + r_3\beta x + g + r_3 + r_2\beta$$

Proof. According to Lemma 1, as well as Theorem 1 in paper [4] where the solution for a general system of generalised pseudo-Boolean functional equations is given, we have conditions (4.2) and (4.2) and the explicit form of the solution (4.3).

Let us consider a functional equation of alternative functions of the form

$$a\frac{\partial f_\alpha}{\partial x} + b\frac{\partial f_\beta}{\partial y} + c\frac{\partial^2 f_{\alpha\beta}}{\partial x \partial y} = g, \tag{5}$$

where a,b,c, and g are known alternative functions from L_2^2 into L_2, and f is an unknown alternative function L_2^2 into $L_2, \alpha, \beta \in L_2$.

$$\begin{aligned} a(x,y) &= a_1x + a_2xy + a_3y + a_4 \\ b(x,y) &= b_1x + b_2xy + b_3y + b_4 \\ c(x,y) &= c_1x + c_2xy + c_3y + c_4 \\ g(x,y) &= g_1x + g_2xy + g_3y + g_4, \quad (x,y) \in L_2^2 \end{aligned} \tag{6}$$

where $a_1, a_2, a_3, a_4, \ldots, g_1, g_2, g_3, g_4$ are constants from L_2.

For various values of α and β from L_2 and for the given functions a, b, c and g there are four different functional equations of alternative functions which have the form (5).

In our further work, we shall consider the following equation

$$F : a\frac{\partial f_0}{\partial x} + b\frac{\partial f_1}{\partial y} + c\frac{\partial^2 f_{01}}{\partial x \partial y} = g \tag{7}$$

Theorem 1. *A functional equation (7) has a solution if and only if the conditions*

$$g_4 = 0 \tag{8.1}$$

$$\begin{aligned} a_4b_4 &= 1 \\ a_4 + b_4 + c_4 &= 1 \\ a_1 + b_1 + c_1 &= 0 \end{aligned} \tag{8.2}$$

$$\begin{aligned} (a_2 \; &+ \; b_2 + c_2)(a_3b_3 + a_3b_4 + a_4b_3 + 1) = 0 \\ (a_3 \; &+ \; b_3 + c_3)(a_3b_3 + a_3b_4 + a_4b_3 + 1) + a_3 + b_3 + c_3 = 0 \end{aligned}$$

are satisfied. Then, all functions f that are solutions are determined by the formula

$$f(x,y) = c_0 + X + Y\frac{\partial x_1}{\partial y}, c_0 \in \{0,1\}, \text{ where} \tag{9}$$

$$X = \frac{\partial f_0}{\partial x} \quad and \quad Y = \frac{\partial f_1}{\partial y}$$

$$\begin{aligned} X &= g[(\frac{\partial a_0}{\partial x} + a + b + c)\frac{\partial b_0}{\partial x} + b + bc + (b+c)](\frac{\partial b_0}{\partial x} + b)\frac{\partial g_1}{\partial y} \\ &+ b(\frac{\partial a_1}{\partial y} + a)\frac{\partial g_0}{\partial x} \\ Y &= g[(\frac{\partial b_0}{\partial x} + a + b + c)\frac{\partial a_1}{\partial y} + a + ac] + (a+c)(\frac{\partial a_1}{\partial y} + a)\frac{\partial g_0}{\partial x} \\ &+ a(\frac{\partial b_0}{\partial x} + b)\frac{\partial g_1}{\partial y} \end{aligned} \tag{10}$$

The explicit form of solution (9) is

$$\begin{aligned} f(x,y) = c_0 + g[&(a + c + b_3y + b_4)(a_1x + a_2x + a_3 + a_4 + a + ac)] \\ &+(b+c)(a_2x + a_3 + a_2xy + a_3y)(g_1x + g_2xy) + a(b_1x + b_2xy) \\ &+(g_2x + g_3 + g_2xy + g_3y) + (g_1x + g_2x + g_3 + g_4)[a_3 + a_4 + b_1x \\ &+b_2x + b_3 + b_4 + c_1x + c_2x + c_3 + c_4) + (b_1x + b_2x) + b_1x + b_2x \\ &+b_3 + b_4 + (b_1x + b_2x + b_3 + b_4)(c_1x + c_2x + c_3 + c_4)] \\ &+(b_1x + b_2x + b_3 + b_4) + (a_1 + a_2x)(g_1x + g_2x), \end{aligned} \tag{11}$$

c_0 is a constant from L_2.

Proof. If we find the partial derivatives $\frac{\partial F_0}{\partial x}$, $\frac{\partial F_1}{\partial y}$ and $\frac{\partial^2 F_{01}}{\partial x \partial y}$ of a functional equation (7), then we get the following system of functional equations:

$$\begin{aligned} a\frac{\partial f_0}{\partial x} &+ b\frac{\partial f_1}{\partial y} + c\frac{\partial^2 f_{01}}{\partial x\partial y} = g \\ a\frac{\partial f_0}{\partial x} &+ \frac{\partial b_0}{\partial x} + \frac{\partial f_1}{y} + (b + \frac{\partial b_0}{\partial x} + c)\frac{\partial^2 f_{01}}{\partial x\partial y} = \frac{\partial g_0}{\partial x} \\ \frac{\partial a_1}{\partial y}\frac{\partial f_0}{\partial x} &+ b\frac{\partial f_1}{\partial y} + (a + \frac{\partial a_1}{\partial y} + c)\frac{\partial^2 f_{01}}{\partial x\partial y} = \frac{\partial g_1}{\partial y} \\ \frac{\partial a_1}{\partial y}\frac{\partial f_0}{\partial x} &+ \frac{\partial b_0}{\partial x}\frac{\partial f_1}{\partial y} + (a + b + \frac{\partial b_0}{\partial x} + \frac{\partial a_1}{\partial y} + c)\frac{\partial^2 f_{01}}{\partial x\partial y} = \frac{\partial^2 g_{01}}{\partial x\partial y} \end{aligned} \tag{12}$$

Denote by M and M' the matrix and the augmented matrix of system (12), respectively.The considered system (12) has a solution for $\frac{\partial f_0}{\partial x}and\frac{\partial f_1}{\partial y}$, if rang

M = rang $M' = 3$ and

$$(13) \qquad (a) \quad \left(\frac{\partial a_1}{\partial y} + a\right)\left(\frac{b_0}{\partial x} + b\right)(c + a + b) = 1$$

$$(b) \quad \frac{\partial g_0}{\partial x} + \frac{\partial g_1}{\partial y} + \frac{\partial^2 g_{01}}{\partial x \partial y} + g = 0$$

From these two conditions, when appropriate partial derivatives are found, follow conditions (8.1) and (8.2).

There remains to prove that function f is of the form (9). First, system (12) has to be solved. From the given conditions

$$X = \frac{\partial f_0}{\partial x}, \quad Y = \frac{\partial f_1}{\partial y}$$

is obtained. Finally according to Lemma 2 the solution of equation (7) has the explicit form (11).

Example. We shall give a solution for a functional equation of alternative functions of the form (7), i.e.

$$a\frac{\partial f_0}{\partial x} + b\frac{\partial f_1}{\partial y} + c\frac{\partial^2 f_{01}}{\partial x \partial y} = g,$$

where

$$\begin{aligned} a(x,y) &= x + xy + 1 \\ b(x,y) &= xy + 1 \\ c(x,y) &= x + 1 \\ g(x,y) &= x + xy, \quad (x,y) \in L_2^2 \end{aligned}$$

The constants from these functions satisfy conditions (8.1) and (8.2). If the values of these constants are substituted in (11), then

$$f(x,y) = c_0 + x + xy, (x,y) \in L_2,$$

c_0 is a constant from L_2^2, is the form of the solution.

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REZIME

O JEDNOJ KLASI FUNKCIONALNIH JEDNAČINA DRUGOG REDA ALTERNATIVNIH FUNKCIJA

U radu su dati potrebni i dovoljni uslovi za postojanje rešenja jedne klase funkcionalnih jednačina drugog reda alternativnih funkcija, kao i sama njena rešenja u eksplicitnom obliku.

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