Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 23, 1 (1993), 199 - 218

Review of Research
Faculty of Science
Mathematics Series

SPACE AND TIME IN THE APPARATUS OF INFINITESIMAL CALCULUS

Aleksandar Nikolić
Faculty of Technical Sciences, University of Novi Sad
Trg Dositeja Obradovića 6, 21000 Novi Sad, Yugoslavia

Abstract

The paper deals with the historic aspect of the role of space and time in Newton's discovery of infinitesimal calculus. The possible influence of time in alchemy on Newton's work is also discussed.

AMS Mathematics Subject Classification (1991): 01A45
Key words and phrases: infinitesimal calculus, space, time, motion, limit, alchemy.

1. Introduction

The publication of Newton's greates work, *Philosophiae Naturalis Principia Mathematica* (The Principles), and his general law of gravity in 1687 was the event which marked the development of science and human thought not only in the 17th century, but for a longer period, actually all the way to Einstein, whose general theory of relativity marks the present time. The predominant feature of the century in which The Principles were developed, the century of geniuses as Whitehead called it, was the complete mathematization and mechanization of the macrocosmos–space and microcosmos–Earth and man. These tendencies led up to the suppressed importance of perception and sensible truths and to the substitution of the world of quality for the world of quantity. Thus the search for measurable elements in the observed phenomena and the relation between these measures and physical quantities became the matters of primary importance. Up to the very beginning of the 16th century the predominant view of the world was the organic view based on the authorities of Aristotle and the church. Its principal aim was the observation of natural phenomena and understanding their importance. In the 16th and 17th century this organic, animistic, spiritual understanding of the world was replaced by a new, entirely different view of the world, the world as a machine. Descartes ideas that man should believe only in what has been approved by intellect and confirmed by expirience, and that the only truth about physical objects is that which has been derived by the clarity of mathematical proofs from statements generally known and accepted as truthful, were the opinions which precisely defined the goal of the organizers of this new system of the world. In creating the conditions which enabled scientists to mathematically describe the world, the contributions of Kepler and Galilei were of the utmost importance. Galilei postulated shape, number and motion – important qualities which can be measured and quantified and, by performing his ingenious experiments in 1638, he discovered the law of falling bodies, while Kepler, who studied Tyche Brahe's astronomic tables with the description of the orbit of Mars, succeeded in empirically deriving the three basic laws of motion of the planets in 1609. Newton used these two discoveries in formulating the general laws of motion of all planets in the solar system. Similarly, Newton's discovery of infinitesimal calculus could not have been possible without his direct or indirect knowledge of the work of Wallis, Fermat, Cavalieri, Galilei and Barrow as well as numerous other mathematicians and philosophers on determining instantaneous velocity, direction of tangent line, area of plane figures, volume of rotating bodies and on studying the nature of infinitely small values. Thanks to his tremendous ability of integration and generalization of facts, Newton came to revolutionary conclusions about the inverseness of the process of determining the area under the curve and determining the direction of the tangent of the given curve. In this way he established the basis for the further development of integral and differential calculus.

2. Absolute and relative aspects of space and time

After his Definitions, Newton in Scholium defines matter, time, space and motion. In these definitions he makes a difference between the absolute and relative aspects of these notions. The absolute aspect of space and time is a frame in which matter is found and in which all motions take place. He defines space and time in the following way: "Absolute, true, and mathematical time, of itself, and from its own nature, flows, equably without relation to anything external, and by another name is called duration: relative, apparent, and common time, is some sensible and external (whether accurate or unequable) measure of duration by the means of motion, which is commonly used intead of true time; such as an hour, a day, a month, a year... Absolute time, in astronomy, is distinguished from relative, by the equation or correction of the apparent time. For the natural days are truly unequal, though they are commonly considered as equal, and used for a measure of time; astronomers correct this inequality in order to measure the celestial motions by a more accurate time. It may be, that there is no such thing as equable motion, whereby time can be accurately measured. All motions may be accelerated and retarded, but the flowing of absolute time is not liable to any change. The duration or perseverance of the existence of things remains the same, whether the motions are swift or slow, or none at all: and therefore this duration ought to be distinguished from what are only sensible measures thereof; and from which we deduce it, by means of the astronomical equation. The necessity of this equation, for determining the times of a phenomenon, is evinced as well from the experiments of the pendulum clock, as by eclipses of the satellites of Jupiter... Absolute space, in its own nature, without relation to anything external, remanins always similar and immovable. A relative space is some movable dimension or measure of the absolute spaces; and which is commonly taken for immovable space; such is the dimension of a subterraneous, and aerial, or celestial space, determined by its position in respect of the earth. Absolute and relative space are the same in figure and magnitude; but they do not remain always numerically the same. For if the earth, for instance, moves, a space of our air which relatively and in respect of the earth remains always the same, will at one time be one part of the absolute space into which the air passes; at another time it will be another part of the same, and so, absolutely understood, it will be continually changed."[1]

[1] [15] p. 6 – 8.

The distinction between the two aspects of time is equivalent to Plato's eternity and time as the picture of eternity. Similarly to Plato, who derives measures for all events from eternity, Newton's absolute time is the duration from which every natural – relative time is derived. This relative time is the relative measure of the constant and uniform duration of absolute, real, and mathematical time measured by cyclical motion. Therefore, Newton's time is neither a scholastic measure of motion nor Descartes' duration of things, but has its own nature and would exist even if the physical world did not. Conceived in this way as a continuous, uniform and infinite flow of mutually identical instants, absolute time can be related to space by comparing these instants of time to points of a straight line or to the uniform motion of a body along the line due to inertion. Also, relative time is linked to space by using as a measure of duration the sector of a circle covered by the moving hand of a clock. Although Newton is considered to have made the final breakthrough towards the purely mathematical idea of linear time, this example shows that cyclic presentation of time duration was still present. The same is apparent from Newton's 1675 letter to Henry Oldenburg in which he writes about nature as an incessant cyclic worker who creates liquids from solids, firm things from evaporable ones, evaporable from firm, delicate from coarse, and coarse from delicate.[2] The letter also shows Newton's belife in the constant flow of transmutation in nature, the topic which will be discussed later in this paper.

According to Newton, space as well as time, exists independent of matter, and would exist even if the world did not. The physical world with bodies in motion is in absolute space as well as in absolute time. In regard to sequence everything is in time, and in regard to order and position everything is in space. To distinguish absolute from relative space, which to our senses seem the same, Newton introduces the difference in number emphasising the uniqueness of absolute space and plurality of relative space. A body is claimed to be in space if it occupies a place, relative or absolute, depending on which space this place is part of. Places are defined using their distance from immovable body and then measuring the motion in relation to that place. Because absolute motion, motion in relation to space, cannot be defined, as opposed to relative motion which is always in relation to other bodies, we use relative place and motion instead of absolute ones. Newton defined absolute motion as a body's movement from one absolute place to another absolute place, while relative motion is defined as motion from one

---

[2] [24] p. 23.

relative place to another. Absolute motion can be defined if observed in an immovable place, while relative ones are defined in relation to movable place. Newton considered rest and motion as having the same value in regard to their existence and, consequently, placed them on the same level. Therefore he considered motion as well as rest as a state and not a process, as it had been concieved throughout ancient times all the way to Galilei. In this way the bodies could move even without any force or cause initiating their movement. However, he considered only uniform and straight–line motion as a state, since, as Coyré wrote, no other, and certainly not cyclic or rotational motion, even if it were uniform, could be considered as such, although it seems that rotational motion can hold just as well, and perhaps even better than straight one, which – it is our experience – is terminal usually rather soon.[3] Contrary to Newton, the Greeks studied only constant speed motion (without acceleration), and took as a suitable model of such uniform motion the eternal cyclic celestial motion. Not surprisingly, they viwed time, in close relation to motion, in its cyclic shape. So when uniform straight–line motion was accepted as a basic form of motion, time, too took a straight–line form, i.e. it took a straight line for a corresponding geometrical–space model. This time model, being a physical model of Euclidian space, could also fit with the universal cosmic coordinate system of there dimensional Euclidian geometry.[4] In a coordinate space of this kind it was easy to quantitatively define different sorts of flux – the path of the moving point in respect to velocity and time. As a consequence, the graphs of different functions could be obtained and, similarly, the direction of the tangent at a certain point and the area under the path–curve could be determined. It was on this mathematical notion of isochronal and homogeneous time, which, with the infinite, eternal, immovable and invariable space, constitutes the framework in which all motions (changes), either uniform or accelerated, take place, that Newton based his calculus of fluxions. Perhaps the crucial influence on the discovery of the calculus of fluxions was that of Newton's predecessor and teacher Isaac Barrow, not only for his explicit comparison of time to a straight line, but also because of the idea of vanishing magnitude, which Barrow marked with "a" and "e", and which Newton also used, marking them with "o" and with "ox" and "oy".

---

[3] [14] p. 205.

[4] [20] p. 162.

3. Space and time in the calculus of fluxions

Barrow believed in the analogy of space and time, for as time does not imply an actual existence, but only the capacity or possibility of the continuance of existence; so does space express the capacity of a magnitude, contained in it. The origin of Newton's idea of absolute time can also be traced down to Barrow's work *Lessons in Geometry* (1916) where he says that time does not imply motion, "...as to its absolute and intrinsic nature any more than does rest. The quantity of time, in itself, depends not on either of them; for whether thing move on, or stand still; whether we sleep or wake, time flows perpetualy with an equal tenor."[5] However, although time does not imply motion it does enable its measurung, and, similarly, duration of time cannot be observed without motion. Barrow claimed that in the same way as we measure space by a certain magnitude and then use that measured space for measuring other magnitudes commensurable with it, so is time used as a mediator in measuring motion. In this way time is a measure of motion. Barrow frequently discussed the analogy between time as a mathematical concept and line. "Again, because time, as has been shown, is a quantum uniformly extended, all whose parts correspond, either proportionably to the respective parts of an equal motion, or to the parts of spaces moved through with an equal motion; it may therefore be very aptly represented to our minds by any magnitude alike in all its parts; and especially by the most simple ones, such as a straight or circular line; between which and time there happens to be much likeness and analogy. For as time consists of parts all together similar, it is reasonable to consider it as a quantity endowed with one dimension only...We therefore shall always express time by a right line."[6] This is the first direct formulation of the concept of linear (geometrical) time. Barrow also compared a line which consists of points, or is a trace of a moving point, to time which consisits of strung moments or is a continual flow (flux) of a moment. Similarly, Newton, certainly under Barrow's influence, expressed the idea that instants of absolute time create a continual string similar to the points of a geometrical line. In this way both Barrow and Newton found it necessary to explain the nature of instantaneous velocity. Barrow was still imprecise and hesitated between an atomistic and an infinitesimalistic concept of time, and he wrote: "To every instant of time, or indefinitely small particle of time, (I say instant or

---

[5] [2] p. 35.
[6] [2] p. 15.

indefinite particle, for it make no difference whether we suppose a line to be composed of points or of indefinitely small linelets; and so in the same manner, whether we suppose time to be made up of instants or indefinitely minute timelets); to every instant of time, I say, there corresponds some degree of velocity, which the moving body is considered to possess at the instant."[7] The vagueness about the nature of the smallest part of time is extended to vagueness about the smallest unit of space. Namely, the area under the curve determined by time and velocity which represents a covered distance can, according to Barrow, be represented as a sum of straight lines drawn perpendicular in every point of the time line (time axis) and the length of which corresponds to velocity at that instant, but also as a sum of very narrow rectangles if the time axis is concieved as consisting of infinitely short time intervals. Thus this problem is reduced to that of the limits of the sum of an infinitely large number of surface elements. Through his calculus of fluxions, the essence of which can eventually be reduced to time vanishing in a moment, Newton developed the notion of limits, bringing it close to its present–day conception established by Cauchy in the 19th century. Barrow realized the connection between the problem of defining the direction of tangential velocity and the problem of area on the one hand and time on the other, but he did not emphasize the inverseness of these problems. And that was precisely what Newton did.

4. Mathematical time and infinitesimal calculus

For the first time Newton explained his main mathematical ideas on infinitesimal calculus in The Principles, where the basic concepts of the three developmental phases of his conception of the essence of infinitesimal calculus are found. Each of the three phases is dealt with saparately in the works written before The Principles were published, but they, like The Principles were published much later, after 1700.

The first approach to the problem of determining the direction of tangent and the inversal problem of determining the area under the curve by means of infinitesimals is explained in a monograph entitled *De analysi per aequationes numero terminorum infinitas,* written in 1669, and published only in 1711. The Second is the approach through fluxions and fluents given in the book *Methodus fluxionum et serierum infinitarum,* dated 1671 and

[7] [2] p. 38.

published in 1736, and the third approach through prime and ultimate ratios is given in his *Tractatus de quadratura curvarum,* written in 1676, and published in 1704.

The first approach is characterized by geometrical and analytical infinitesimals within the binomial theorem. Newton here applied the idea of infinitely small rectangular areas as infinitesimals of space, and calculated the area under the given curve, or the curve, if the area below it is given. This procedure is illustrated using the following example with the given area z

$$z = ax^n \tag{1}$$

where n is a whole number or a fraction. Newton called the infinitely small increment of the independant variable x moment, and, like James Gregory in his *Geometriae pars universalis,* (1668) marked it with " o ". Next, the area bounded by the required curve, the x–axis, the y–axis and the ordinate in abscissa $x + \mathrm{o}$ was denoted by $z + \mathrm{o}y$ where $\mathrm{o}y$ is the moment (infinitesimal increment) of the area z. Then

$$z + \mathrm{o}y = a\,(x + \mathrm{o})^n \tag{2}$$

By subtracting (1) from (2) we obtain

$$\mathrm{o}y = a\,(x + \mathrm{o})^n - ax^n$$

Then by applying the binomial theorem

$$\mathrm{o}y = a\,\mathrm{o}\,nx^{n-1} + a\,\mathrm{o}\,\mathrm{o}\,\frac{n(n-1)}{2}x^{n-2} + a\,\mathrm{o}\,\mathrm{o}\,\mathrm{o}\,\frac{n(n-1)(n-2)}{6}x^{n-3} + \,\dots$$

When both the left and right side are divided through by " o " we obtain

$$y = anx^{n-1} + a\,\mathrm{o}\,\frac{n(n-1)}{2}x^{n-2} + a\,\mathrm{o}\,\mathrm{o}\,\frac{n(n-1)(n-2)}{6}x^{n-3} + \,\dots$$

and, by neglecting all terms containing " o ", Newton obtained

$$y = anx^{n-1}$$

which represents the curve under which is the given area $z = ax^n$. The procedure can also be viewed inversely, i.e. if the curve $y = anx^{n-1}$ is given, the area below it is $z = ax^n$.

Here Newton first determined the momentary rate of change of one variable (z) in relation to other (x), which is, in fact, the differentiation and then calculated the area by the inverse process of the finding rate of change which is the indefinite integral of the curve (function) y. Most important in this process was his presupposition of spatial infinitesimals which, for the purpose of determining area, he did not add, but regarded them as momentary increase of area in the process of infinitesimal increase of the independant variable x. This means that in this procedure he did not use the limiting value of the sum but, as it has been mentioned, supposed that the area is obtained through the inverse process from calculating the rate of change. In this way, in fact, the fundamental theorem of infinitesimal calculus of summation (integration) as the inverse process of differentiation was expressed for the first time. This procedure was the origin of the second phase of Newton's work on infinitesimal calculus, in which he presented the cinematic concept of fluxions and fluents, the concept in which time plays the dominant role. Namely, if the ordinate y is regarded as the velocity of the increasing area and abcissa x as time, this gives a real infinitesimalistic concept of calculating area, where the product of y and the interval on $x-$axis (the interval which reperesents infinitely small time increment) gives the infinitely small segment of the area under the curve. The whole area is calculated by adding these moments. But, unlike his predecessors who calculated the area by precisely such a procedure, Newton calculated the entire area only from instantaneous rate of change in one point. Through this procedure he did not state clearly why, in what way, and under which circumstances he had left out the terms which contain " o ", the fact that caused a lot of criticism.

In *De quadratura curvarum* Newton writes the following on fluxions and fluents: "...I consider mathematical quantities in this place not as consisting of very small parts, but as described by a continual motion. Lines are described and thereby generated, not by apposition of parts, but the continued motion of points; superficies by the motion of lines; solids by the motion of superficies; angles by the rotation of the sides; portions of time by continued flux."[8] He considered the magnitudes increasing in equal time intervals to be greater or smaller depending on whether they increase at greater or smaller velocity and sought to find a method for determining magnitudes according to velocity or rate of increase which caused them. He used the term fluxion for such velocity or rate of change and fluent for nascent quality. In this way he found the method of fluxions which he used in determining the area under the curve.

[8] Quotation from [13] p. 363.

If the fluents, as nascent quantities, were marked by x and y, fluxions were marked by $\dot{x}$ and $\dot{y}$. Higher order fluxions or fluxions of fluxions were marked by $\ddot{x}$ and $\ddot{y}$, etc. Fluents, the fluxions of which were x and y were marked by x' and y', and their fluents i.e. fluents of fluents by x'' and y''.

The two basic problems of analysis are expressed in the following way, using the terminology of the method of fluxions:

1. The relation between fluents being given, to determine the relation of their fluxions. This, in fact, is the differentiation of function of variables which are time dependent.

2. By means of given equation containing fluxion of quantities, to find the relation of those quantities–fluents. This is the problem of integrating the differential equation.

By kinematic terminology, these two problems would correspond to finding the velocity of a moving body if the distance covered in space is given, and finding covered distance in space if the velocity is given, respectively. Both problems are inseparable from the notion of time. Therefore, fluents are closely related to time but, instead of being simply functions of time as independant variable, they are related to their fluxions as rate of changing and so, indirectly, to time.

Newton observed the curve $f(x,y)=0$ as a place of intersection of two lines in motion, one vertical and one horizontal. In this way the coordinates (x and y) are functions of time and they mark the position of the vertical and horizontal line. Motion can be understood as a composition of horizontal motion with velocity vector $\dot{x}$ and vertical motion with velocity vector $\dot{y}$. Tangential velocity vector is a sum of these two vectors, so the direction of tangent line results from relation $\dot{y}/\dot{x}$. If both of these motions are uniform, the covered distances are proportional to their velocities, and if the motions are arbitrary, we consider the infinitely short time interval (" o ") in which any motion is essentially uniform. In this way $\dot{x}$ and $\dot{y}$ are velocities found by differentiating x and y with respect to time t $(dy/dt,\ dx/dt)$, and their relation $(\dot{y}/\dot{x})$ is equal to dy/dx.

If " o " is an infinitely small time interval then xo and yo are infinitely small increments or moments – instantaneous changes of fluents. "These quantities I here consider as variable and indetermined in increasing or decreasing as it were by a continual motion or flow (flux); and I understand their momentary increments or decrements by the name of moments; so that the increment may be esteemed as added or positive moments, and decrements as subtracted or negative ones...Moments are to be understood as the just nascent beginning of finite quantities,"[9] writes Newton in The Principles. It follows from this that the curve is considered in status nascendi and moments (change of velocity) in vanishing time. In order to see how Newton determined the relation between $\dot{y}$ and $\dot{x}$ let us consider the fluent $y = x^n$. Newton first forms

$$y + \dot{y}\mathrm{o} = (x + \dot{x}\mathrm{o})^n \tag{3}$$

and then expands the right hand side according to the binomial theorem, subtracts $y = x^n$ from (3), and divides the equation by " o ", neglects all terms containing " o " and obtains $\dot{y} = nx^{n-1}\dot{x}$ or $\dot{y}/\dot{x} = nx^{n-1}$, which, in modern notation is equivalent to equations

$$\frac{dy}{dt} = nx^{n-1}\frac{dx}{dt} \quad \text{or} \quad \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = nx^{n-1}$$

and after dviding through by dt we obtain $dy/dx = nx^{n-1}$ which is the present–day notation of differentiation.

In order to establish infinitesimal processes as precisely as possible, Newton introduced and developed in his third approach in *De quadratura curvarum* the theory of limiting ratios as the theory of prime and ultimate ratios of change. While in the first two approaches to infinitesimal calculus Newton used certain infinitesimals in such a way that any change or any motion consists of infinitely small changes, the time of infinitely short time intervals, space of infinitely small spatial units–moments, in this approach he tried to avoid infinitely small quantities by making moments (increments and decrements) the beginnings or ends of nascent and evanescent quantities. Again, the method will be demonstrated by using the function $y = x^n$. Newton marked the increment of the independent variable x, i.e. time, by " o ". Since, further in this procedure, he dealt with the ratio of functions of a single variable, he did not take "ox" as moment, regardless of the fact that this notation is more in accord with the calculus of fluxions. Now $(x + \mathrm{o})^n$ is the increment of the function x^n. The problem is to find the fluxion of the fluent x^n in the state of nascent. Developing $(x + \mathrm{o})^n$ according to the

[9] [15] p. 249.

binomial theorem we obtain

$$x^n + n \circ x^{n-1} + \frac{n(n-1)}{2} \circ \circ x^{n-2} + \dots$$

Unlike the previous procedure from the *Method of fluxions* where in the equation with two variables he simply neglected the terms containing " o " since "o" was something that vanished or nothing, as is the case in Berkeley's lemma,[10] here the emphasis is on function of one variable. After the precise formulation of the liniting processes this kind of criticism was elminated. Newton compared increment of x (" o ") to increment of x^n ($n \circ x^{n-1} + \frac{n(n-1)}{2} \circ \circ x^{n-2} + \dots$) and obtained the proportion

$$\frac{\circ}{n \circ x^{n-1} + \frac{n(n-1)}{2} \circ \circ x^{n-2} + \dots}$$

and after dividing through by " o " he obtained

$$\frac{1}{nx^{n-1} + \frac{n(n-1)}{2} \circ x^{n-2} + \dots}$$

This ratio can also be understood as the ratio of time and velocity increment. At this point Newton supposes that " o " vanishes and therefore he gets the ultimate ratio of vanishing increment $1/nx^{n-1}$, which is, in fact, the limit of the ratio of the change. The ultimate ratio of the evanescent increment is the same as the prime ratio of the nascent quantity. Newton defined ultimate ratios most precisely in The Principles: "Ultimate ratios in which quantities vanish, are not, strictly speaking, ratios of ultimate quantities, but limits to which the ratios of these quantities decreasing without limit, approach, and which, though they can come nearer than any given difference whatever, they can neither pass over nor attain before the quantities have diminshed indefinitely."[11] This is, in fact, the reply to Berkley's criticism of

[10]In *The Analyst*, 1734, Georges Berkeley attacked Newton's approach to analysis by means of flux and vanishing quantities, which F.Cajori called a turning point moment in the history of mathematical thought in Great Britain ([6]). In *The Analyst* Berkeley formulated the following lemma:"If in a demonstration an assumption is made, by virtue of which certain conclusions follow, and if afterward that assumption is destroyed or rejected, then all the conclusions that had been reached by the first assumption must also be destroyed or rejected." By this lemma he attacked Newton's increment " o " which is arbitrary different from 0 (when the formula is developed according to the binomial theorem) or equal to 0 (when terms containing it are neglected).

[11][15] p. 39

Newton's fluxions and ultimate ratios of vanishing quantities, which according to Berkley were neither ultimate quantities nor zero, but "the ghosts of departed quantities". In the same way Newton defined the ultimate velocity as the velocity "with which the body is moved, neither before it arrives at its last place, when the motion ceases, nor after, but at the very instant when it arrives". This ultimate velocity is, in fact, instantaneous velocity. However, even at this stage Newton did not eliminate infinitesimals completely, since the ratio $1/nx^{n-1}$ is obtained by transformation of a quantitative ratio of time and velocity into a ratio of instant as time infinitesimals and moment of kinematic infinitesimals. There "o" flows like time passing through different values and at one, last instant has the value of zero. But then, if time increment is zero, space increase is also zero. Therefore the ratio $0/0$ is pointless and because of this there is no instantaneous velocity. Berkley relaised this and criticised this concept of differential calculus as well. Newton himself relaised that there still were several problems with this kind of explanation, so he allowed a certain definite quantity for velocity increment and, writing about ultimate ratio he said that, strictly speaking, they were not the ratios of ultimate quantities. All this shows a great deal of speculation and absence of precise understanding of the concepts of infinity, continuity, and real numbers, which basically is the essence of limit. Newton came close to this idea and, according to Boyer, had he devoted more time to explaining the elements appearing in the definition of ultimate ratio, infinitesimal calculus would have been established a century before Cauchy.[12]

It seems that the method of fluxions was the most appropriate to Newton's way of thinking and this method is the closest to Cauchy's concept of derivative as the limit of the ratio

$$\lim_{\Delta x \to o} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Namely, in the example $y = x^n$, Newton finds the difference between $f(x + \Delta x) - f(x)$ by subtracting y from $y + o\dot{y}$, i.e. subtracting x^n from $(x + o\dot{x})^n$, because of $y = x^n$ and $y + o\dot{y} = (x + o\dot{x})^n$. In this way he obtains

$$o\dot{y} = (x^n + nx^{n-1}\, o\,\dot{x} \ldots) - x^n$$

If both sides of the equation are divided through by ox we obtain the ratio

$$\frac{\dot{y}}{\dot{x}} = \frac{(x^n + nx^{n-1}\, o\,\dot{x} \ldots) - x^n}{o\dot{x}}$$

[12] [4] p. 196.

in which the expression in brackets corresponds to $f(x+\Delta x)$ and $o\dot{x}$ corresponds to Δx. By further dividing through by ox we obtain

$$\frac{\dot{y}}{\dot{x}} = nx^{n-1} + \frac{n(n-1)}{2}x^{n-2} \circ \dot{x} + \ldots$$

At this stage Newton neglects all terms containing "o", which is equivalent to allowing Δx to approach 0, and the result is $\dot{y}/\dot{x} = nx^{n-1}$ which corresponds to $dy/dx = nx^{n-1}$. This is again very close to the method of ultimate ratios, the fact that would have been more prominent had the relation of increment x^n and x been observed and not other way round, as Newton did. It can be concluded that there is no substantial difference between the three phases of Newton's discovery of infinitesimal calculus. If we said more precisely that neglected are all the terms containing $o\dot{x}$ and not only " o ", i.e. $o\dot{x}$ approaches 0, this would be very close to Cauchy's conception of limiting process where Δx approaches 0. There is only one, last step to be taken towards rigid mathematical conception of infinitesimal calculus, namely its deliverance of motion and therefore of velocity, space, and time, in other words reducing it to a purely mathematical notion.

5. Time in alchemy and infinitesimal calculus

Another notion has an important role in Newton's conception of infinitesimal calculus, and that is the idea of constant change. or transmutation. This suggests examining another aspect of the influence of time on Newton's discovery of differential and integral calculus. namely the specific role of time in alchemy and alchemic processes. Newton devoted a great deal of his creative work to this field, the fact which has often been neglected in the history of science.

The origin of his knowledge is manifold. Kepler and Galilei had the primary influence on Newton's scientific – astronomic, physical, and mathematical ideas, while the system–philosophical influence came from the other side, from Descartes and Francis Bacon. In the 17th century two basic scientific–philosophical systems, opposite to each, other were present; one preferred the analytical–deductive method advocated by Descartes, another the empirical–inductive method advocated by Bacon. Each of the two methods was, in proving and describing certain phenomena and notions, at the same time superior and inferior to the other. Newton sensed this correctly and, in The Principles, presented a harmonious synthesis of both methods emphasizing that neither experiments devoid of system basis and explanation, nor deduction from primeaval principles devoid of experimental proof can lead to reliable and complete theory. By surpassing both Bacon's systematic experimenting and Descartes non–experimental mathematical analysis and by integrating the two, he developed an entirely mathematized formulation of mechanicistic view of the world which was to form the basis for the scientific development in the two centuries to follow. It is clear that Newton's words that if he had seen a lttle farther than others it was because he stood on the shoulders of giants, were legitimate and truthful. Is it possible, however, that some of the giant ideas were neglected in the scientific and historical analysis of Newton's classical science? Had there been something in his work that was, in the fantastic development in the period of scientific revolution and later, omitted and forgotten? There is certainly more in Newton's thinking than mathematics and experience. First and foremost, evident is his clear insight in to the limits of a purely mechanical view of the world, as well as his understanding that the only real truth is at the same time scientific, philosophical, mystical, and perceptive and that it requires the entirety of thought and knowledge. It was only in the last fifteen years, mostly through the works of B.J.T. Dobbs, R.S. Westfall, J.E. McGuire, and P.M. Rattansi, that science historians and scientists themselves came to seriously study Newton's unpublished works and came to the conclusion that the final formulations of matter, motion, time, forces, in the universe have origins in mechanistic philosophy, theology, metaphysics, mysticism, hermetism, alchemy and stoicism.

It has been known that Newton read books of ancient Greek myths, the works of many philosophers of ancient times, as well as those of E. Ashmole, J. Dee, T. Norton, H. Kunrath, R. Lull, R. Fludd, J. Boehme, Paracelsus, well known Renaissance alchemists, hermetists, and Rosicrucians. Up till recently no one even suspected the role these works had in Newton's classic scientific discoveries. Dobbs's publication of the results of her reserach and analysis of Newton's unpublished alchemic papers, which, after the auction at Sotheby's had been collected by John Maynard Keynes between 1937 – 39, shed a new light on Keynes's words that "Newton was not the first of the age of reason. He was the last of the magicians, the last of the Babylonians and Sumerians, the last great mind which looked out on the visible and intellectual world with the same eyes as those who begun to build our intellectual inheritance rather less than 10000 years ago."[13] Regardless of the fact that

[13] John Mynard Keynes, Newton the Man, Royal Society Newton Tercentenary Cele-

these esotheric roots of Newton's ideas were despised, energetically rejected or ignored and passed over in silence, Westfall claims that the presence of alchemy and hermetic philosophy in Newton's workis beyond doubt, and the fundamental question for Newtonian scholarship is the mutual interaction of the two scientific and philosophical systems.[14] He interprets Newton's work after 1675 as one long attempt to integrate alchemy and the mechanical philosophy.

After reading about these ideas, Newton was, till the end of his life, preocupied with Transmutation which he understood as the process of disorganiztion and reorganization throygh which matter is reduced, revived, and developed into new forms. The alchemic agent responsible for this process is the universal fermental virtue. Its primary role is to putrefact a uniform whole and destroy it into a chaos, because "nothing can be changed from what it is without putrefaction...Nothing can be generated or nourished (but of puterefied matter)."[15] Explained in this way putrefaction completely corresponds to the first phase of Great Alchemic Work (Opus Magnum) – Transmutation which leads to Materiae Primae and symbolizes death, i.e. reduction of matter to a shapeless state that marks premordial cosmic chaos in which all parts of matter are similar and capable of being reduced into any form. Newton explained his belief in the possibility of transmutation as early as in The Principles:"...Any body can be transformed into another, of whatever kind, and the intermediate degrees of qualities can be induced in it ...matter falling to Earth from the tails of comets might be condensed into all types of Earth substances." This belief is perhaps most explicitely expressed in *Opticks* (1704), in the Question 30: "The changing of bodies into light, and light into bodies, is very comfortable to the cause of nature, which seems delighted with transmutations...". Next he gives examples of transmutations in nature finishing with the following words:"...and among such various and strange transmutations, why may not nature change bodies into light, and light into bodies?" In the next, 31st question, when he mentions the effect of Aqua Fortis and Aqua Regis on various metals, Newton's real alchemist magician nature is revealed in rhe most beautiful way.

But what is the role of man, alchemist in the Great Work, in that in-

---

brations, 15-19 july, 1946,quotation from [10] p. 13.

[14] Westfall R., Newton and Hermetic Tradition, Science, Medicine and Society in the Rennaisance, 1972, p. 186.

[15] Newton, *On the Nature of Acids* (1691), quotation from B.J.T. Dobbs, Newton's Alchemy and his Theory of Matter, ISIS, 1982, 73 (269) p. 519.

finitely long Transmutational process of nature? His role is to accelerate the rhythm of time and infinite process of nature and to insert it into a short human life. The desire for transforming processes which take place over geological time scales into processes which take place over a human time scale has its origin in prehistoric times when people belived in the Mother Earth in whose insides all metals "grew" and "ripened", metals which, had they stayed in the ground long enough, perhaps infinitely long, would have turned into gold as the finest metal. The extraction of ore was like a "premature birth" and by his acts the alchemist changes the minerals and metals into gold by an accelerated procedure. By taking upon himself the role of nature the alchemist acts in behalf of time.

Being a great alchemist Newton knew of the role and importance of time and wove it into his system of the world as one of the basic notions. The absolute time defined in Scholium as infinite duration, completely independent from any motion observable by the senses in the physical world, corresponds to eternal transmutation of "lower" metals into their "highest" form – gold. The relative time which can be measured and derived through motion by the senses corresponds to the time that a man, an alchemist needs to do the same transmutation by accelerated procedure and which can be confirmed by its products at a certain stage. Therefore, motion as a constant state is analoguos to eternal transmutation and, consequently, the velocity of motion is analoguos to that of transmutation. In his constant desire to accelerate it and to beat time, the alchemist introduces fire and heat in his experiments. This acceleration of the procedure corresponds to the acceleration of body in motion. Thus, the "growth" of metals in the ground into gold as well as the work of an alchemist trying to achieve the same goal, are facts determined only by the duration of time. Of course, any transmutation can be "interrupted" at any moment and, by putrefaction the newly created product can again be turned back into shapeless black Preliquid. The possibility for reverseness is analoguos to finding the flux i.e. the quantity in the state of nascent from the fluent i.e. the generated quantity and vice versa. The whole analogy, in itself a little metaphysical, finds its main support in the fact that both alchemical processes and Newton's view of infinitesimal calculus are based on the change in time flow. In this way Newton's involvement in alchemy, too, could be supposed to have had an important role in the way he conceived the inverseness of the processes of integrating and differentiating.

Acknowledgement: The author wishes to thank Ivana Mirović, English language lector at the Faculty of Technical Sciences, for her help with the translation of this article from Serbo–Croat into English.

References

[1] Arsenijević M., Prostor, Vreme, Zenon, Biblioteka Teka Filozofsko društvo Srbije i Grafički zavod Hrvatske, Beograd & Zagreb, 1986.

[2] Barrow I., Geometrical Lectures, Open Court, Chicago, London, 1916. 16.

[3] Berkli G., Rasprava o pitanjima ljudskog saznanja, BIGZ, Beograd, 1977.

[4] Boyer C. B., The History of the Calculus and its Conceptual Development, Dover, New York, 1949.

[5] Boyer C. B., A History of Mathematics, John Wiley & Sons, New York, 1968.

[6] Cajori F., Discusion of Fluxion: From Berkley to Woodhouse, The American Mathematical Monthly, vol XXIV, no.4. April 1917.

[7] Čapek M., The Concepts of Space and Time – Their Structure and Their Development, Reidel Publishing Company, Dordrecht Holland and Boston – U.S.A., 1976.

[8] Dadić Ž., Razvoj matematike, Školska knjiga, Zagreb, 1975.

[9] Dijksterhuis E. J., The Mechanization of the World Picture, Princeton, 1986.

[10] Dobbs B. J. T., The Foundations of Newton's Alchemy, Cambridge University Press, 1975.

[11] Edwards C. H. Jr., The Historical Development of the Calculus, Springer–Verlag, New York, Heidelberg, Berlin, 1979.

[12] Jammer M., Concepts of Space, Harvard University Press, Cambridge, Massachusetts, 1969.

[13] Kline M., Mathematical Througt from Ancient to Modern Times, Oxford University Press, New York, 1972.

[14] Koare A., Naučna revolucija, Nolit, Beograd, 1981.

[15] Sir Isaac Newton's Mathematical Principles of Natural Philosophy and his System of the World, Translated into English by Andrew Motte in 1729. The translation revised and supplied with an historical and explanatory appendix by Florian Cajori, Berkley, 1946.

[16] Newton I., Opticks or a Tratise of the Reflections, Refactions, Inflections & Colours of Light, based on the fourth edition London 1730, Dover, New York, 1952.

[17] Unpublished scientific papers of Isaac Newton. A selection from the Portsmouth Collection in the University Liberary Cambridge. Chosen, edited and translated by A. Rupert Hall and Marie Boas Hall, Cambridge University Press, 1958.

[18] Isaac Newton's Papers and Letters on Natural Philosophy and Related Documents, Edited with general introduction by I. Bernard Cohen, assisted by R. E. Schoefield, Harvard University Press, Cambridge, Massachusetts, 1958.

[19] Nikolić A., O Njutnovom znanju, Zbornik radova simpozijuma Njutnova Philosophia Naturalis – nastanak i prevazilaženje, Kragujevac, 1987.

[20] Pavlović B. U., Filozofija prirode, Naprijed, Zagreb, 1978.

[21] Reichenbach H., The Philosophy of Space & Time, Dover, New York, 1958.

[22] Time & Sciences, (zbornik) UNESCO, Paris, 1979.

[23] Whitrow G. J., The Natural Philosophy of Time, Thomas Nelson and Sons Ltd., London and Edinburgh, 1960.

[24] Whitrow G. J., Šta je vreme, Prosveta, Beograd, 1985.

[25] Whitrow G. J., Mathematical Time and its Role in the Development of the Scientific World–view, (vidi [22]).

REZIME

VREME I MESTO U INFITEZIMALNOM RAČUNU

Rad se bavi istorijskim aspektom uloge prostora i vremena u Njutnovom otkriću infinitezimalnog računa. Takodje je iznesen mogući uticaj vremena u alhemiji u Njutnovom radu.

Received by the editors December 13, 1990