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ON RICCI-RECURRENT SEMI-DECOMPOSABLE RIEMANNIAN SPACES WITH A VANISHING SCALAR CURVATURE

Nevena Pušić[1]
Institute of Mathematics, University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

The paper treats the problem of Ricci-recurrent semi-decomposable (pseudo)Riemannian spaces, whose scalar curvature is a constant and consequently, its reccurence vector need not to be a gradient of scalar curvature logarythm. A number of properties of such spaces and also, following the classification of well-are known spaces with the semi-decomposable metric, some properties of their Ricci tensors are given.

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1. Introduction

A Riemannian space is said to be semi-decomposable if there exists a co-ordinate system in which the metric of the space can be expressed in this way:

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$$(1)\qquad ds^2 = g_{ij}dx^i dx^j = g_{ab}dx^a dx^b + \sigma a_{\alpha\beta}dx^\alpha dx^\beta$$

where gab and the function σ depend only on coordinates $x^a(a = 1, \ldots, q)$ and $a_{\alpha\beta}$ depend only on coordinates $x^\alpha(\alpha = q+1, \ldots, n)$. Indices $_{i}, _{j}, _{k}$ take all the values from 1 to n (n is the dimension of the space).Directly from expression (1), one can conclude that a locally semi-decomposable space V_n consists of two mutually orthogonal families of subspaces:V_q and V_{n-q}. Any member of the V_q-family is defined by $g_{ab}(x^c)$, while x^α are fixed and, reversely, while x^a are fixed, $a_{\alpha\beta}(x^\gamma)$ are coefficients of the first fundamental form of the space V_{n-q} .

Let us consider a semi-decomposable Riemannian space V_n and its coordinate system which enables expression (1). In such a coordinate system, just these Christoffel symbols are not equal to zero:

$$(2)\qquad \Gamma^a_{bc} = \Gamma^a_{(1)bc};\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{(2)\beta\gamma};\Gamma^\alpha_{a\beta} = \frac{1}{2\sigma}\sigma_{,a}\delta^\alpha_\beta;\Gamma^a_{\beta\gamma} = -\frac{1}{2}g^{ab}\sigma_{,b}a_{\beta\gamma}$$

$\Gamma^a_{(1)bc}$ denotes a Christoffel symbol of a subspace V_q, with respect to metric g_{ab}; $\Gamma^\alpha_{(2)\beta\gamma}$ is a Christoffel symbol with respect to the metric $g_{\alpha\beta}$; a comma in front of an index denotes a covariant derative with respect to metric (1) (in the case of a scalar function σ, it is equal to its partial derivative). Every object of the space V_q carries the index $_{(1)}$ and every object of the space V_{n-q} carries the index$_{(2)}$.

Using relation (2), we can get that just the following components of curvature tensor of V_n

$$(3)\qquad \begin{gathered}(i)R_{abcd} = R_{(1)abcd};\\ (ii)R_{\alpha\beta\gamma\delta} = \sigma R_{(2)\alpha\beta\gamma\delta} + \frac{1}{4}\Delta_1\sigma(a_{\alpha\gamma}a_{\beta\gamma} - a_{\alpha\delta}a_{\beta\gamma});\\ (iii)R_{\alpha ab\beta} = \sigma\, T_{ab}a_{\alpha\beta}\end{gathered}$$

are not equal to zero, where

$$(4)\qquad \begin{gathered}\Delta_1\sigma = g_{ab}\sigma_{,a}\sigma_{,b},\\ T_{ab} = -\frac{1}{2\sigma}[\sigma_{,a.b} - \frac{1}{2\sigma}\sigma_{,a}\sigma_{,b}]\end{gathered}$$

The space V_n is said to be decomposable if in (1), $\sigma = 1$ (or some other constant). Then, $V_n = V_q \times V_{n-q}$ holds locally. A space is said to be Ricci-recurrent if its Ricci tensor is recurrent i.e. if it satisfies the relation:

$$R_{ij,k} = \kappa_k R_{ij}. \tag{5}$$

where (κ_k) is a vector field. We presume that our Ricci-recurrent space is not Ricci-parallel i.e. it does not hold that

$$\kappa_k = 0 \text{ for every k or } R_{ij} = 0 \text{ for every } i, j.$$

If the scalar curvature R of the space V_n is a non-constant function, then the the vector (κ_k) is a gradient vector field and, moreover, $\kappa_k = \frac{\partial ln|R|}{\partial x^k}$. Such a case has been investigated in paper [3].

2. Results

Here, we shall suppose that the scalar curvature R of a semi-decomposable Ricci-recurrent Riemannian space is a constant. Then, we can immediately prove

Theorem 1. *If the scalar curvature of a Riemannian space V_n is a constant and if the space is Ricci-recurrent (not Ricci-parallel), then the scalar curvature vanishes.*

Proof. Since the Riemannian space is Ricci-recurrent, then relation (5) holds and, since it is not Ricci-parallel, then $\kappa_k \neq 0$ for at least one value of k. Since R is a constant, then, for at least one value of k there holds

$$R_{,k} = (R_{ij}g^{ij})_{,k} = R_{ij,k}g^{ij} = \kappa_k R_{ij}g^{ij} = \kappa_k R = 0$$

and, consequently, $R = 0.\square$

Choosing a semi-decomposable Ricci-recurrent Riemannian space with constant scalar curvature as an object of consideration, we have chosen a semi-decomposable Ricci-recurrent space with vanishing scalar curvature.

In a semi-decomposable Riemannian space, just the following components of the Ricci tensor

$$(6) \qquad R_{ab} = R_{(1)ab} + (n-q)T_{ab}$$

$$R_{\alpha\beta} = R_{(2)\alpha\beta} + (\frac{\Delta_1\sigma}{4\sigma}r + \sigma T_{ab}g^{ab})a_{\alpha\beta}$$

do not vanish. r stands for the integer $q+1-n$.

Taking into account that the space V_n is Ricci-recurrent, we can obtain the following relations:

$$(7) \qquad (i)R_{(1)ab,c} = \kappa_c R_{(1)ab} + \kappa_c(n-q)T_{ab,c}$$

$$(ii)R_{ab,\alpha} = \kappa_\alpha R_{ab} = 0$$

$$(iii)\kappa_c R_{(2)\alpha\beta} = [\frac{\partial}{\partial x^c}(\frac{\Delta_1\sigma}{4\sigma})r + \sigma_{,c}T_{ab}g^{ab} + \sigma T_{ab,c}g^{ab} - \kappa_c(\frac{\Delta_1\sigma}{4\sigma}r + \sigma T_{ab}g^{ab})]a_{\alpha\beta}$$

$$(iv)R_{(2)\alpha\beta;\rho} = \kappa_\rho[R_{(2)\alpha\beta} + (\frac{\Delta_1\sigma}{4\sigma}r + \sigma T_{ab}g^{ab})a_{\alpha\beta}]$$

$R_{(1)ab}$ denotes Ricci tensor of the metric g_{ab}; $R_{(2)\alpha\beta}$ denotes the Ricci tensor of the metric $a_{\alpha\beta}$; the symbol $.c$ denotes the covariant differention with respect to the metric g_{ab} and the symbol *rho* denotes the covariant differention with respect to the metric $a_{\alpha\beta}$.

Now, we can prove following

Theorem 2. *If the scalar curvature of a semi-decomposable Ricci-recurent Riemannian space vanishes, then each of subspaces V_{n-q} has a constant scalar curvature.*

Proof. By formula (6) and by the fact that the scalar curvature of V_n vanishes, we have the relation

(8) $$O = R_{(1)} + 2(n-q)T_{ab}g^{ab} + (n-q)\frac{\Delta_1\sigma}{4\sigma^2} + \frac{1}{\sigma}R_{(2)}$$

$R_{(1)}$ denotes the scalar curvature of the space $V-q$ and $R_{(2)}$ denotes the scalar curvature of the space V_{n-q}. Since the left-hand side of equality (8) vanishes and every member of the right-hand side depends only on variables x^a, then $R_{(2)}$ also depends on the variables x^a only. But $R_{(2)}$ is an object of the inner geometry of $V_n - q$. Any member of subspace family V_{n-q} could be got by fixing the values of $x^a s-$ consequently, by fixing $R_{(2)}$. Taking into account this fact, we can conclude that on any V_{n-q}, $R_{(2)}$ is a constant.□

Contracting the formula (7) (iv) by $a^{\alpha\beta}$, we get

(9) $$R_{(2);\rho} = \kappa_\rho[R_{(2)} + (n-q)(\frac{\Delta_1\sigma}{4\sigma}r + \sigma T_{ab})] = 0$$

Taking into account (7)(ii), we can see that there are two possible cases:

$$\text{I}\ \ \kappa_\alpha = 0$$

$$\text{II}\ \ R_{(2)} + (n-q)(\frac{\Delta_1\sigma}{4\sigma}r + \sigma T_{ab}g^{ab}) = 0 \text{ and} R_{ab} = 0$$

We shall analize now both of these cases.

I If $\kappa_\alpha = 0$ for every α, then $\kappa_c \neq 0$ for at least one c and then, according to (7)(iii), V_{n-q} is an Einstein space. Since we have proved that it has a constant scalar curvature, it is Ricci-parallel.

Let us fix that c, for which $\kappa_c \neq 0$ and contract (7)(iii) by $a^{\alpha\beta}$. Then we get

(10) $$\kappa_c[R_{(2)} + (n-q)(\frac{\Delta_1\sigma}{4\sigma}r + \sigma T_{ab}g^{ab})] =$$
$$(n-q)[\frac{\partial}{\partial x^c}(\frac{\Delta_1\sigma}{4\sigma})r + \sigma_{,c}T_{ab}g^{ab} + \sigma T_{ab.c}g^{ab}]$$

We saw that $R_{(2)}$ depends only on variables x^a, but on every V_{n-q} it is a constant. Now, we shall consider $R(2)$ as a function of the whole family V_{n-q}. If we denote

$$(11)\qquad \varphi = (n-q)(\frac{\Delta_1\sigma}{4\sigma}r + \sigma T_{ab}g^{ab})$$

then (10) takes the form

$$\kappa_c(R_{(2)} + \varphi) = \varphi_{,c}$$

Diferentiating this form of (10) covariantly, we get finally

$$(12)\qquad (\kappa_{c,d} - \kappa_{d,c})(R_{(2)} + \varphi) + \kappa_c R_{(2),d} - \kappa_d R_{(2),c} = 0$$

From (12), we can conclude that, if $\kappa_\alpha = 0$ for every α, then (κ_k) is a gradient vector field if and only if it is proportional to the gradient of the function $R_{(2)}$ or it the function $R(2)$ is an absolute constant.

II Let $\kappa_\alpha \neq 0$, for at least one α; then, by (7)(ii)

$$(13)\qquad R_{ab} = 0 \text{ and, by (9)}$$

$$(14)\qquad R_{(2)} = -(n-q)(\frac{\Delta_1\sigma}{4\sigma}r + \sigma T_{ab}g^{ab})$$

which is a constant on every space V_{n-q}.

We can show that in case II, $\kappa_c = 0$, for every c. Suppose that $\kappa_c \neq 0$, for at least one c. Then, according to (7)(iii), V_{n-q} is an Einstein space. Since (14) is fulfiled, then

$$R_{(2)\alpha\beta} = -(\frac{\Delta_1\sigma}{4\sigma}r + \sigma T_{ab}g^{ab})a_{\alpha\beta}$$

and, according to (6), $R_{\alpha\beta} = 0$. But, in the case II, $R_{ab} = 0$, and, consequentlly, our space V_n is Ricci-flat. Our assumption that V_n is non-trivially Ricci-recurrent proves that $\kappa_c = 0$, for every c.

Let us consider now the whole family of V_{n-q}s. Varying the member of this family, we also vary $R_{(2)}$ and change the fixed points of various V_q. As $\kappa_c = 0$ for any c, we get from (7)(iii);

$$\frac{\partial}{\partial x^c}(\frac{\Delta_1\sigma}{4\sigma})r + \sigma_{,c}T_{ab}g^{ab} + \sigma T_{ab,c}g^{ab} = 0$$

or

$$\frac{\partial}{\partial x^c}\left(\frac{\Delta_1\sigma}{4\sigma}r + \sigma T_{ab}g^{ab}\right) = 0 \tag{15}$$

which means, according to (14), that $\frac{\partial}{\partial x^c}R_{(2)} = 0$ and $R_{(2)}$ is an absolute constant.

Besides, as $R_{ab} = 0$, then, by (6)

$$R_{(1)ab} = -(n-q)T_{ab}$$

We proved

Theorem 3. *If the scalar curvature of a semi-decomposable Ricci-recurrent Riemannian space vanishes, then*

a) V_{n-q} is an Einstein space with a constant scalar curvature;or
b) All the V_{n-q}s have the same scalar curvature (which is constant), the Ricci tensor of the space V_q depends only on function σ and the function σ satisfies the partial differential equation

$$(r+1)\frac{\Delta_1\sigma}{\sigma} - 2\Delta_2\sigma = const,$$

where $\Delta_1\sigma$ and $\Delta_2\sigma$ are Beltrami operators of the first and second kind, respectively.

Our investigation was not very informative about the spaces $_q$; actually, our assumptions about V_n gave too few properties for V_q. If we want V_q to be an Einstein space, a space of a constant scalar curvature, a Ricci-recurrent space or something similar, we have to involve some additional analytical conditions, as we have done in [3].

3. Appendix

In [2] some kinds of semi-classification of semi-decomposable spaces was given. Actually several groups were named, well-known for their other properties, which happen to be semi-decomposable in some coordinate system.

There are spaces of constant, positive or negative, sectional curvature, subprojective spaces of Kagan and spaces of concircular geometry.

If the sectional curvature of a space is constant, then it is proportional to its scalar curvature, and, according to Theorem 1, we have

Corollary 1. *A space of constant (positive or negative) sectional curvature cannot be Ricci-recurrent.*

Now, we shall consider subprojective spaces of Kagan. By the result of Kručkovič ([1]), we can divide subprojective spaces in to two groups

$$(1.A)\ \text{I}\ ds^2 = (dx^1)^2 + \sigma(x^1)(ds_1)^2(x^2,\ldots,x^n)$$

where $(ds_1)^2$ is an $(n-1)$-dimensional metric of the constant sectional curvature-"basic case", and

$$(2.A)\ \text{II}\ ds^2 = 2dx^1dx^2 + \sigma(x^1)ds_1(x^3,\ldots,x^n)$$

where $(ds_1)^2$ is an $(n-2)$-dimensional Euclidean metric (with a vanishing sectional curvature)-"exstraordinary case". For the basic case, there holds ([1]):

$$(3.A)\ R_{jk} = f(x^1)g_{jk} + \varphi(x^1)\delta^1_j\delta^1_k; (j,k = 1,\ldots,n),$$

where

$$f(x^1) = \frac{K(n-2)}{\sigma} - \frac{\sigma''}{2\sigma} - \frac{\sigma'^2}{4\sigma^2}(n-3)$$

and

$$\varphi(x^1) = \frac{2-n}{2}[(1n\sigma)'' + \frac{2K}{\sigma}].$$

It is easy to see that the components $R_{\alpha\beta}(\alpha,\beta = 2,\ldots,n)$ **are** recurrent; but they are proportional to components $R_{(2)\alpha\beta}$andV_{n-q} is, in the basic case, an Einstein space with a constant scalar curvature. By this fact, the recurrence of $R_{\alpha\beta}$ is trivial. R_{11} is recurrent if $\varphi(x^1) = \lambda f(x^1)and\lambda \neq 0$(for$\lambda =$

0,V_n is a space of a constant sectional curvature). For the extraordinary case,

$$(4.A)\ R_{(2)ab}=0, R_{(2)\alpha\beta}=0, \Delta_1\sigma=0, T_{ab}g^{ab}=0, T_{ii}=-\frac{1}{2\sigma}[\sigma"-\frac{1}{2\sigma}\sigma'^2]$$

and

$$(5.A)\ R_{ii}=-(n-2)T(x^1), \text{the rest of} R_{jk}=0.$$

From (5.A) we can see

$$(6.A)\ R_{jk}=\varphi(x^1)\delta_j^1\delta_k^1$$

where

$$\varphi(x^1)=\frac{2-n}{2\sigma}[\sigma"-\frac{1}{2\sigma}\sigma'^2]$$

The space V_n will be of a constant scalar curvature if $\varphi(x^1)=0$. Now, we have proved

Theorem 4. *A subprojective space V_n is Ricci-recurrent if the function σsatisfies*

$$\frac{K(n-2)}{\sigma}-\frac{\sigma"}{2\sigma}-\frac{(\sigma')^2}{4\sigma}(n-3)=\alpha\frac{2-n}{2}[(1n\sigma)"+\frac{2K}{\sigma}]$$

$$\frac{K(n-2)}{\sigma}-\frac{(\sigma")^2}{4\sigma^2}(n-3)\neq[\frac{K}{\sigma}-\frac{1}{4}(1n\sigma)'^2](n-1)$$

for the basic case and

$$\sigma\neq(Ax^1+B)^2A, B=const$$

for the extraordinar case.

Finally, we shall consider the spaces of a concircular geometry. These are spaces with a conformal transformation transforming geodesic circles into geodesic circles. Analytically, this means that there exists a function ρ satisfying

$$(7.A)\ \rho_{i,j} - \rho_{j,i} = \psi g_{ij}.$$

By a lemma from [4], this means that the metric of such a space can be expressed as

$$(8.A)\ ds^2 = (dx^1)^2 + \sigma(x^1)(ds_2)^2(x^2, \ldots, x^n).$$

As we can see, the basic case of a subprojective space is a special case of a space with concircular geometry, with V_{n-q} of a constant sectional curvature. Here we do not know anything about the geometry of V_{n-q}. We can have numerous assumptions about its geometry and it can be Ricci-recurrent. But a space of concircular geometry satisfies the condition

$$\rho^t R_{ts} = (n-1)(\psi\rho_s - \psi, s).$$

Differentiatiating the last equation one more time and taking into account that the space is Ricci-recurrent, we get

$$\psi R_{s\tau} = (n-1)[\kappa_\tau(\psi, s - \psi\rho_s) + \psi, r\rho_s + \psi^2 g_{rs} + \psi\rho_\tau\rho_s - \psi_{,\tau,s}].$$

Since the Ricci tensor of a Riemannian space is symmetric, we have

Theorem 5. *A space of concicular geometry is Ricci-recurrent if and only if it satisfies the condition*

$$\kappa r\psi, s - \psi\kappa_\tau\rho_s + \psi_{,\tau}\rho_s = \kappa_s\psi_{,\tau} - \psi\kappa_s\rho_\tau + \psi_{,s}\ rho_\tau,$$

where ψ is the function from (7.A), ρ_s is the concircular vector field and κ_τ is the recurrence vector.

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REZIME

O RIČI-REKURENTNIM POLUDEKOMPONOVANIM RIMANOVIM PROSTORIMA ČIJA JE SKALARNA KRIVINA JEDNAKA NULI

U radu se razmatra problem Riči-rekurentnih poludekomponovanih (pseudo) Rimanovih prostora čija je skalarna krivina konstantna. Dat je niz osobina ovakvih prostora. Takodje, za neke poznate prostore koji imaju poludekomponovanu metriku date su osobine njihovih Ričijevih tenzora.

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