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l^p-PERTURBATIONS OF A LINEAR DIFFERENCE EQUATION

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Abstract

The paper deals with a general linear difference equation and gives conditions under which its linear perturbation preserves the same l^p-affiliations

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1. Introduction

We will start with two well known theorems concerning ordinary differential equations.

Theorem A. (Dini - Hukuhara [1, p.36]). *If* $q_i(t)$, $t \in [0, \infty)$ *are measurable functions and* $q_i(t) \in L[0, \infty), i = 0, \cdots, n-1$, *if* C_i *are real numbers such that the equation*

$$Ky \equiv y^{(n)} + C_{n-1}y^{(n-1)} + \cdots + C_0 y = 0$$

has all solutions bounded in $[0,\infty)$, *then, also, the equation*

$$K_L y \equiv Ky + q_{n-1}(t)y^{(n-1)} + \cdots + q_0(t)y = 0$$

has all solutions bounded.

Theorem B. (Weyl [2]) *Consider the equation*

$$y''(t) + p(t)y(t) = 0, \; t \geq 0, \tag{1}$$

along with the perturbed linear equation

$$y''(t) + (p(t) + q(t))y(t) = 0, \; t \geq 0, \tag{2}$$

where both $p(t)$ *and* $q(t)$ *are locally integrable on* $[0,\infty)$. *Then, the following Weyl alternative holds: if all solution of (1) are (not) in* $L^2[0,\infty)$ *and* $q(t) \in L^2[0,\infty)$ *then all solutions of (2) are (not) in* $L^2[0,\infty)$.

Weyl's alternative result was extended by Patula and Wong [3] to include arbitrary $L^p[0,\infty)$-perturbation $q(t)$.

Also, Mironov [4] extended this theorem to the case of the delay-differential equation, while Wyrwinska [5] considered some more general classes of perturbations. Wong [6] obtained related results for a restrictive class of $2n$-th order differential equations. All these results were generalized and improved in paper [7] to the general n-th oreder linear differential equation and corresponding linear and non-linear perturbations with or without retardations.

The purpose of this paper is to extend some of the results from [7] to the case of a general n-th order linear difference equation of the form

$$Dy \equiv y(n+m) + p_{n-1}(m)y(n+m-1) + \cdots + p_0(m)y(m) = 0, \tag{3}$$

$m = 0, 1, \cdots$ and corresponding linear perturbation

$$D_L y \equiv Dy + q_{n-1}(m)y(n+m-1) + \cdots + q_0(m)y(m) = 0, \; m = 0, 1, \cdots \tag{4}$$

2. Preliminaries

In this section we will introduce some notations and list well-known results which will be used in the sequel.

The symbol Δ is a forward difference operator i.e. $\Delta y(m) = y(m+1) - y(m)$. By a solution of a difference equation we mean a real sequence $\{y(m)\}$, $m = 0, 1, \cdots$ satisfying it, and troughout this paper we will usually refer to a solution $\{y(m)\}$, $m = 0, 1, \cdots$ simply as a solution y.

Let $\mathcal{D}$ be any of the difference operators D, D_L, K or K_L. Then we denote

$$S^k(\mathcal{D}) = \{\Delta^k y \, : \, \mathcal{D}y = 0\} \text{ for } k = 0, 1, \cdots, n-1.$$

So $S(\mathcal{D})$ denotes the set of all the solutions of (3), (4), (7) or (8), $S^1(\mathcal{D})$ denotes the set of all the first differences of solutions of these equations, etc. Further, let y_i $i = 1, \cdots, n$ be a linearly independent solution of (3).

Our first result is a discrete form of the variation of the parameters formula.

Lemma 1. *Suppose $F(u, v)$ is a continuous function on R^2 and y is the solution of $Dy = F(m, y(m))$. Then*

$$\Delta^j y(m) = \sum_{i=1}^{n} C_i \Delta^j y_i(m) + \sum_{s=1}^{m-1} F(s, y(s)) \Delta^j_m G(s, m), \; j = 0, 1, \cdots, n-1,$$

where

$$G(s,m) = \frac{\begin{vmatrix} y_1(s+1) & \cdots & y_n(s+1) \\ \vdots & & \vdots \\ y_1(s+n-1) & \cdots & y_n(s+n-1) \\ y_1(m) & \cdots & y_n(m) \end{vmatrix}}{\begin{vmatrix} y_1(s+1) & \cdots & y_n(s+1) \\ \vdots & & \vdots \\ y_1(s+n) & \cdots & y_n(s+n) \end{vmatrix}} \equiv \frac{W(s,m)}{D(s+1)}$$

$D(s)$ is Casorati's determinant and for it we have:

Lemma 2. (Heymann's theorem [8, p.354]). *Casorati's determinant $D(s)$ satisfies the linear difference equation of the first order*

$$D(s+1) = (-1)^n p_0(s) D(s).$$

We also need the following, obvious

Lemma 3. *If the sequence $U \in l^p$, $p \geq 1$, then it is also true for* ΔU.

The next result is the discrete generalization of the fundamental inequality of Gronwall and Bellmann, as far as some non-linear variants of that inequality

Lemma 4. (Willet and Wong [9]). *Suppose that $V(n), W(n)$ and $U(n+1)$, $n = 0, 1, \cdots$, are non-negative sequences of numbers with $V(0) = W(0) = 0$ and that U_0 and p are constants such that $U_0 > 0$ and $p \geq 0$, $p \neq 1$. Then the inequality*

$$U(n+1) \leq U_0 + \sum_{j=0}^{n} V(j)U(j) + \sum_{j=0}^{n} W(j)U^p(j), \ n = 0, 1, \cdots$$

implies that

$$e(n)U(n+1) \leq [U_0^{1-p} + (1-p)\sum_{j=0}^{n} W(j)e^{1-p}(j)]^{\frac{1}{1-p}}, \ n = 0, 1, \cdots$$

where

$$e(n) = \prod_{j=0}^{n}(1 + V(j))^{-1}, \ n = 0, 1, \cdots$$

The next elementary inequalities will be used in the sequel.

Lemma 5. (Hardy, Littlewood and Polya [10 p.26]). *Let $a_i \geq 0$ for $i = 1, \cdots, m$. Then*

$$\sum_{i=1}^{m} a_i^p \leq m^{1-p}(\sum_{i=1}^{m} a_i)^p, \text{ for } 0 \leq p \leq 1,$$

and

$$m^{p-1}\sum_{i=1}^{m} a_i^p \geq (\sum_{i=1}^{m} a_i)^p \text{ for } p > 1.$$

3. Results

Our first result is an extension of a discrete version of Weyl's alternative theorem

Theorem 1. *Let $S(D) \subseteq l^2$ and $q_i \in l^\infty$ for $i = 0, \cdots, n-1$. If exist M such that $\prod_{m=0}^{s} |p_0(m)| \geq M > 0$ for $s = 0, 1, \cdots$ then $S(D_L) \subseteq l^2$.*

Proof. From Lemma 1 we conclude that all the solutions of (4) can be represented by

(5) $$y(m) = \sum_{i=1}^{n} C_i y_i(m) + \sum_{i=1}^{n} y_i(m) \sum_{s=0}^{m-1} D_i(s+1) \frac{(-1)^{n+i} f(s)}{D_0 \prod_{k=0}^{s} p_0(k)},$$

$m = 0, 1, \cdots$ where D_i denotes the Casoratian of $y_1, \cdots, y_{i-1}, y_{i+1}, \cdots, y_n$ $D_0 = D(0)$ and

$$f(s) = -\sum_{i=0}^{n-1} q_i(s) y(i+s) = \sum_{i=0}^{n-1} \overline{q_i}(s) \Delta^i y(s).$$

Obviously $\overline{q_i}(s)$ are linear combinations of $q_i(s)$ which implies that $\overline{q_i(s)} \in l^\infty$ too.

Taking into account the fact that $G(s, m) = 0$ for $s = m-n+1, \cdots, m-1$ we obtain

$$\Delta^k y(m) = \sum_{i=1}^{n} C_i \Delta^k y_i(m) + \sum_{i=1}^{n} \Delta^k y_i(m) + \sum_{s=0}^{m-1} (-1)^{n+i} \frac{D_i(s+1) f(s)}{D_0 \prod_{k=0}^{s} p_0(k)}$$

for $m + 0, 1, \cdots$ and $k = 0, \cdots, n-1$.

According to Lemma 3, $S(D) \subseteq l^2$ implies that $S(D) \subseteq l^\infty$, which gives

$$|D_i(s+1)| \leq K \sum_{j=1 j \neq i}^{n} |y_j(s+1)| \leq K \sum_{j=1}^{n} |y_i(s+1)|$$

for every $i = 1, \cdots, n$ and some constant $K > 0$. So, by (5) we get

$$|\Delta^k y(m)| \leq \sum_{i=1}^{n} |\Delta^k y_i(m)| [K_1 + K \sum_{s=0}^{m-1} \sum_{j=1}^{n} |y_j(s+1)| \frac{|f(s)|}{|D_0| \prod_{k=0}^{s} |p_0(k)|}],$$

for $k = 0, \cdots, n-1$; $m = 0, 1, \cdots$ or

$$|\Delta^k y(m)| \leq K_2 \Phi_k(m) [1 + \sum_{s=0}^{m-1} \Phi_0(s+1) \frac{|f(s)|}{|D_0| \prod_{k=0}^{s} |p_0(k)|}]$$

$k = 0, \cdots, n-1$, $m = 0, 1, \cdots$ where we denote

$$\Phi_k(m) = max\{|\Delta^k u_i(m)| \ : \ i = 1, \cdots, n\}$$

for $k = 0, \cdots, n-1$. The low bound of $|\prod_{m=0}^{s} p_0(m)|$ immediately implies

(6) $\quad |\Delta^k y(m)| \leq K_3 \Phi_k(m)[1 + \sum_{s=0}^{m-1} \Phi_0(s+1)|f(s)|],$

$k = 0, \cdots, n-1$, $m = 0, 1, \cdots$

Hence, by the Cauchy-Schwarz inequality, we derive

$$|\Delta^k y(m)| \leq K_3 \Phi_k(m)[1 + (\sum_{s=0}^{m-1} \Phi_0^2(s+1))^{0.5} (\sum_{s=0}^{m-1} f^2(s))^{0.5}],$$

$k = 0, \cdots, n-1$, $m = 0, 1, \cdots$ which, in view of the fact that $\Phi_0 \in l^2$, implies that

$$|\Delta^k y(m)| \leq K_4 \Phi_k(m)[1 + (\sum_{s=0}^{m-1} f^2(s))^{0.5}],$$

$k = 0, \cdots, n-1$, $m = 0, 1, \cdots$ and

$$|\Delta^k y(m)|^2 \leq K_5 \Phi_k^2(m)[1 + \sum_{s=0}^{m-1} f^2(s)],$$

$k = 0, \cdots, n-1$, $m = 0, 1, \cdots$

Now, taking into account the fact that $\overline{q}_i \in l^\infty$, $i = 0, \cdots, n-1$ we have

$$|\Delta^k y(m)|^2 \leq K_5 \Phi_k^2(m)[1 + \sum_{s=0}^{m-1} |\sum_{i=0}^{n-1} \overline{q}_i(s) \Delta^i y(s)|^2] \leq$$

$$\leq K_6 \Phi_k^2(m)[1 + \sum_{s=0}^{m-1} \sum_{i=0}^{n-1} |\overline{q}_i(s)|^2 |\Delta^i y(s)|^2] \leq$$

$$\leq K_7 \Phi_k^2(m)[1 + \sum_{i=0}^{n-1} \sum_{s=0}^{m-1} |\Delta^i y(s)|^2],$$

$k = 0, \cdots, n-1$, $m = 0, 1, \cdots$, which, after summation from 0 to $n-1$, implies

$$\sum_{k=0}^{n-1} |\Delta^k y(m)|^2 \leq$$

$$\leq K_7\Phi_k^2(m)[1+\sum_{s=0}^{m-1}\sum_{i=0}^{n-1}|\Delta^i y(s)|^2].$$

Define now

$$u(m)=\sum_{i=0}^{n-1}|\Delta^i y(m)|^2,\quad m=1,2,\cdots$$

From the last relation we have

$$u(m)\leq K_7\sum_{k=0}^{n-1}\Phi_k^2(m)[1+\sum_{s=0}^{m-1}u(s)],$$

or

$$\sum_{m=0}^{p}u(m)\leq K_7\sum_{k=0}^{n-1}\sum_{m=0}^{p}\Phi_k^2(m)[1+\sum_{s=0}^{m-1}u(s)].$$

Put $w(p)=\sum_{m=0}^{p}$, from $\Phi_k\in l^2$ we get

$$w(p)\leq K_8[1+\sum_{m=0}^{p}\Phi(m)\sum_{s=0}^{m-1}u(s)]=$$

$$K_8[1+\sum_{m=0}^{p}\Phi(m)W(m-1)]=K_8[1+\sum_{m=0}^{p-1}\Phi(m+1)W(m)]$$

where we denote $\Phi(m)=\sum_{i=0}^{n-1}\Phi_i^2(m)$ and take $u(-1)=0$, which implies that $W(-1)=0$. Using Lemma 4, we get $e(p-1)w(p)\leq K$ which is equivalent to

$$w(p)\leq\frac{K}{e(p-1)}=K\prod_{j=0}^{p-1}(1+\Phi(j+1))<\infty$$

and it implies $\sum_{j=1}^{\infty}\Phi(j+1)<\infty$, because from $\Phi_k\in l^2$ it follows $\Phi\in l$.

Corollary 1. *Let $S(D)\not\subseteq l^2$ and $q_i\in l^\infty$. If $\prod_{m=0}^{s}\geq M>0$ for every $s=0,1,\cdots$ then $S(D_L)\not\subseteq l^2$.*

Proof. Suppose that $S(D_L)\subseteq l^2$ which, by Theorem 1, implies that $S(D)\subseteq l^2$, since D_y can be considered as a linear perturbation of D_L i.e.

$$Dy(m)=D_Ly(m)-\sum_{i=0}^{n-1}q_i(m)y(i+m).$$

Remark 1. In the special case, if $p_0(m) \equiv 1$ and $q_m \equiv 0$, where the operators D and D_L take respectively the forms

(7) $\quad My \equiv y(n+m) + p_{n-1}(m)y(n+n-1) + \cdots + y(m) = 0, \; m = 0, 1, \cdots$

and

(8) $\quad M_L y \equiv My + q_{n-1}(m)y(m+n-1) + \cdots + q_1(m)y(m+1) = 0, \; m = 0, 1, \cdots$

we obtain, from Theorem 1 and Corollary 1, the stronger result as follows.

Theorem 1'. *If $q_i \in l^\infty$ for $i = 1, \cdots, n-1$ then $S(M) \subseteq l^2$ if and only if $S(M_L) \subseteq l^2$.*

The next result is a generalisation of theorem A of Dini and Hukuhara in the case of a difference equation.

Theorem 2. *Let $S(D) \subseteq l^\infty$ and $q_i \in l$ for $i = 0, 1, \cdots, n-1$. If $\prod_{s=0}^{m} |p_0(S)| \geq M > 0$ for every $m = 0, 1, \cdots$ then $S(D_L) \subseteq l^\infty$.*

Proof. In the same way as in the proof of Theorem 1, we obtain relation (6), which, by the fact that $\Phi_k \in l^\infty$, for $k = 0, 1, \cdots, n-1$, immediately implies

$$|\Delta^k y(m)| \leq K_y \Phi_k(m)[1 + \sum_{s=0}^{m-1} |f(s)|] \leq$$

$$\leq K_y \Phi_k(m)[1 + \sum_{s=0}^{m-1} \sum_{i=0}^{n-1} |q_i(s)||\Delta^i y(s)|] \leq$$

$$\leq K_{10}[1 + \sum_{s=0}^{m-1} q(s) \sum_{i=0}^{n-1} |\Delta^i y(s)|],$$

$k = 0, \cdots, n-1$, $m = 0, 1, \cdots$, where we denote

$$q(s) = max\{|q_i(s)| : i = 0, \cdots, n-1\}.$$

The above inequality gives

$$\sum_{k=0}^{n-1} |\Delta^k y(m)| \leq K_{11}[1 + \sum_{s=0}^{m-1} q(s) \sum_{i=0}^{n-1} |\Delta^i y(s)|], \quad m = 0, 1, \cdots.$$

Puting $u(m) = \sum_{k=0}^{n-1} |\Delta^k y(m)|$ we get

$$u(m) \leq K_{11} + K_{11} \sum_{s=0}^{m-1} q(s)u(s), \quad m = 0, 1, \cdots$$

which, by Lemma 4 gives

$$u(m) \leq \frac{K}{e(m-1)} = K \prod_{s=0}^{m-1} (1 + q(s)) < \infty,$$

and it is equivalent with $\sum_{s=0}^{\infty} q(s) < \infty$ or $q \in l$.

Using the same idea as in the Proof of Corollary 1 from Theorem 2 we can prove

Corollary 2. *Let $S(D) \not\subseteq l^\infty$ and $q_i \in l$ for $i = 0, \cdots$, if $\prod_{s=0}^{m} |p_0(s)| \geq M > 0$ for every $m = 0, 1, \cdots$ then $S(D_L) \not\subseteq l^\infty$.*

In the special case of difference operators M and M_L, we obtain a stronger conclusion.

Theorem 2'. *Let $q_i \in l$ for $i = 0, \cdots, n-1$, then $S(M) \subseteq l^\infty$ if and only if $S(M_L) \subseteq l^\infty$.*

Remark 2. Difference equation $My = 0$ is of great importance in practice.

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REZIME

l^p - PERTURBACIJE LINEARNIH DIFERENCNIH JEDNAČINA

U radu se posmatra opšta linearna diferencna jednačina i daju uslovi pod kojima njena linearna perturbacija očuvava istu l^p pripadnost.

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