Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 23, 1 (1993), 135 - 144

Review of Research
Faculty of Science
Mathematics Series

DIFFERENCE SCHEME IN THE FIELD OF MIKUSIŃSKI OPERATORS

Djurdjica Takači
Institute of Mathematics, University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

In the field of Mikusiński operators, $\mathcal{F}$, we consider the linear differential equation of order two with variable coefficients. Using the corresponding discrete analogue for the considered problem, we construct approximate solutions at each point $\lambda_i \in [0,1], i = 1, 2, ...n$. Also, we estimate the error of approximation.

AMS Mathematics Subject Classification (1991): 47A60, 65J10
Key words and phrases: Mikusiński calculus, difference scheme.

1. Introduction

In this paper we consider the following problem

$$(1) \qquad -\frac{\partial^{2+p}u(\lambda,t)}{\partial t^p \partial\lambda^2} + q(\lambda)\frac{\partial^m u(\lambda,t)}{\partial t^i} = f(\lambda),$$

with the appropriate conditions

$$(2) \qquad \frac{\partial^{\mu+\nu}u(\lambda,0)}{\partial\lambda^\mu \partial t^\nu} = 0; \quad \begin{array}{ll} \mu = 0; & \nu = 0,1,...,m-1, \\ \mu = 2; & \nu = 0,1,...,p-1, \end{array}$$

$$(3) \qquad x(0,t) = C_1 t^{p-1}, \qquad x(1,t) = D_1 t^{p-1},$$

where $t \geq 0$, $\lambda \in [0,1]$, $p, m, \in N$, $p \geq m$, while $q(\lambda)$ and $f(\lambda)$ are continuous real valued functions of one variable, and C_1 and D_1 are given constants.

In the field of Mikusiński operators, $\mathcal{F}$, equation (1) with conditions (2) and (3) corresponds to the problem

$$(4) \qquad - s^p u''(\lambda) + q(\lambda) s^m u(\lambda) = f(\lambda), \qquad \lambda \in [0,1]$$

$$(5) \qquad u(0) = C l^p \qquad u(1) = D l^p$$

where s is the differential operator and l is the integral operator, while I is the identical operator. Equation (4) can be written in a form

$$(6) \qquad - u''(\lambda) + Q(\lambda) u(\lambda) = l^p f(\lambda), \quad Q(\lambda) = l^r q(\lambda), \quad r = p - m.$$

The functions $u(\lambda), q(\lambda)$ and $f(\lambda)$ are operatór functions.

An operational function $g(\lambda)$ is continuous ([2]) in a finite (open or closed) interval J if it can be represented as

$$g(\lambda) = q\{g_1(\lambda, t)\}$$

where $g_1(\lambda, t)$ is a continuous function (in the usual sense) in the domain $\Omega = \{(\lambda, t), \lambda \in J, t \geq 0\}$, and q is some operator from $\mathcal{F}$.

An operational function $g(\lambda)$ has a continuous n-th derivative $g^{(n)}(\lambda)$ on an interval J if there exist an operator q_n and a parametric function $\{g_n(\lambda, t)\}$ with the continuous partial derivative $\{\frac{\partial^n}{\partial \lambda^n} g_n(\lambda, t)\}$ in the domain Ω, such that

$$g(\lambda) = q_n\{g_n(\lambda, t)\}, \quad g^{(n)}(\lambda) = \{\frac{\partial^n}{\partial \lambda^n} g_n(\lambda, t)\}.$$

An operational function $g(\lambda)$ is said to be differentiable ([2]) at the point λ_0 if it can be represented as $g(\lambda) = q\{g_1(\lambda, t)\}$, where q is an operator and $g_1(\lambda, t)$ is a parametric function such that the quotient

$$\frac{g_1(\lambda, t) - g_1(\lambda_0, t)}{\lambda - \lambda_0}$$

uniformly tends to the limit for $\lambda \to \lambda_0$ in every finite interval $0 \leq t \leq t_0$.

Let us suppose that the operational function, representing the solution of our problem, $u(\lambda)$ has a continuous fourth derivative ([2]) on the same interval. This implies that the function $u(\lambda)$, has its first, second and third derivative at the point $\lambda_0, \lambda_0 \in [0,1]$. The operational functions $q(\lambda)$ and $f(\lambda)$ are, by supposition, continuous operational functions on the interval $[0,1]$

2. Difference analogue for boundary value problem

Let us denote by

$$h = \frac{1}{n+1}, \quad n \in N, \quad \lambda_j = jh, \quad j = 1, 2, ...n+1,$$

the set $I_h = \{\lambda_j, \quad j = 0, 1, ...\}$ is a grid and

$$u_j = u(\lambda_j) = \{u(\lambda_j, t)\}, \quad q_j = Q(\lambda_j), \quad f_j = f(\lambda_j).$$

In order to approximate $u"(\lambda)$ and $u'(\lambda)$ at the points I_h we shall use the following difference quotients:

(7) $$(Du)(\lambda) = \frac{-u(\lambda - h) + 2u(\lambda) - u(\lambda + h)}{h^2}$$

(8) $$(D^0)u(\lambda) = \frac{u(\lambda + h) - u(\lambda - h)}{2h}.$$

In the field of Mikusiński operators it holds

Lemma 1. *If the operational function $u(\lambda)$ has a continuous fourth derivative on the interval $[0, \lambda_2]$, then there exist points $y_1 \in [0, \lambda_2], y_2 \in [0, \lambda_2]$ and $\lambda \in [0, \lambda_2]$ such that*

(9) $$-u"(\lambda) = (Du)(\lambda) + \frac{h^2 u^{(4)}(y_1)}{12},$$

(10) $$u'(\lambda) = (Du^0)(\lambda) - \frac{h^2 u'''(y_2)}{6}.$$

Proof. Since the function $u(\lambda)$ has continuous fourth derivative, there exists an operator $q \in \mathcal{F}$ such that it holds

$$u^{(4)}(\lambda) = q\{\frac{\partial^4 u(\lambda, t)}{\partial \lambda^4}\}$$

where $\frac{\partial^{(4)} u(\lambda,t)}{\partial \lambda^4}$ is a continuous function in the domain $[0, 1] \times [0, \infty)$. Using the Taylor formula for the functions of two variables and observing the corresponding expression in the field of Mikusiński operators, we have

$$u(\lambda + h) = u(\lambda) + u'(\lambda)h + \frac{1}{2}u"(\lambda)h^2 + \frac{1}{3!}u'''(\lambda)h^3 +$$

$$+\frac{1}{4!}u^{(4)}(y_1)h^4, \qquad y_1 \in (0,1)$$

$$u(\lambda-h) = u(\lambda) - u'(\lambda)h + \frac{1}{2}u"(\lambda)h^2 - \frac{1}{3!}u'''(\lambda)h^3 +$$

$$+\frac{1}{4!}u^{(4)}(y_1)h^4, \qquad y_1 \in (0,1).$$

Using the denotations given in (7), (8) and from previous relations we obtain relations (9) and (10) .

Taking

$$(Du)(\lambda_j) + \frac{h^2u^{(4)}(\theta_j)}{12}, \quad \theta_j \in (0,1),$$

instead of $-u"(\lambda)$ in the problem (5), (6), we obtain the discrete analogue

$$-\frac{u_{j+1}+2u_j+u_{j-1}}{h^2} + q_jv_j = l^kf_j - \frac{h^2u^{(4)}(\theta_j)}{12}, \quad j=1,...,n, \tag{11}$$

$$u_0 = Cl^k, \qquad u_{n+1} = Dl^k. \tag{12}$$

The previous system can be written in the form

$$Au = d + \tau(u) \tag{13}$$

where $A, u, d, \tau(u)$ are corresponding matrices in the field $\mathcal{F}$ such that

$$A = h^{-2}\begin{bmatrix} 2I+q_1h^2 & -I & 0 & \dots & 0 & 0 \\ -I & 2I+q_2h^2 & -I & \dots & 0 & 0 \\ 0 & -I & 2I+q_3h^2 & \dots & 0 & 0 \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ 0 & 0 & 0 & \dots & -I & 2I+q_nh^2 \end{bmatrix}, \tag{14}$$

$$u = \begin{bmatrix} u_1 \\ u_2 \\ . \\ . \\ . \\ v_n \end{bmatrix}, \; d = l^r \begin{bmatrix} f_1 + Ch^{-2} \\ f_2 \\ . \\ . \\ . \\ f_{n-1} \\ f_n + Dh^{-2} \end{bmatrix}, \; \tau(u) = -\frac{h^2}{12}\begin{bmatrix} u^{(4)}(\theta_1) \\ u^{(4)}(\theta_2) \\ . \\ . \\ . \\ u^{(4)}(\theta_{n-1}) \\ u^{(4)}(\theta_n) \end{bmatrix}. \tag{15}$$

Let us consider the system $Av = d$, instead of the system given by (13), where A and d are given by relations (14) and (15) and the values $v_1, v_2, ..., v_n$ are in fact the approximations for $u_1, u_2, ..., u_n$ which represent the solutions of the system (13). We have the following

Theorem 1. *If in equation (6) it holds that* $p \geq m$, *and* $q(\lambda) \geq 0$, *then the solution of the system* $Av = d$ *,where* A *and* d *are given in (14) and (15), can be written as*

$$(16) \qquad v_n = \frac{r_n}{\alpha_n}, \quad v_j = \frac{r_j + h^{-2} v_{j+1}}{\alpha_j}, \quad j = n-1, n-2, ..., 1,$$

where

$$(17) \quad \alpha_1 = h^{-2}(2I + h^2 q_1), \quad \beta_j = \frac{a_j}{\alpha_{j-1}}, \quad \alpha_j = h^2(2 + h^2 q_j) + \beta_j h^{-2},$$

and

$$(18) \qquad r_1 = l^p(f_1 + Ch^{-2}), \quad r_j = f_j l^p - \beta_j r_{j-1},$$
$$r_n = l^p(f_n + Dh^{-2}) - \beta_n r_{n-1}, \quad j = 2, 3, ..., n-1.$$

Proof. We have to prove that $\alpha_j \neq 0$, for $j = 1, 2, ...n$. From relation (17) in the field $\mathcal{F}$ it follows

$$\alpha_1 = b_1 = h^{-2}(2I + h^2 q_1) = h^{-2}(\gamma_1 I + \alpha_{1,c}),$$

$$\beta_2 = \frac{a_2}{\alpha_1} = -\frac{I}{2I + h^2 Q_2} = -\frac{1}{2}\sum_{i=0}^{\infty}(-1)^i(\frac{h^2 q_1}{2})^i.$$

If $p > m$ in equation (6), then the operators $q_1, q_2, ..., q_n$ represent continuous functions, then the infinite series in the previous relation is an operationally convergent series (see [1], pp 180), so we can write

$$\beta_2 = \delta_2 I + \beta_{2,c},$$

where δ_2 is a numerical constant and $\beta_{2,c}$ is an operator representing a continuous function. Also, it holds

$$\alpha_2 = b_2 - \beta_2 c_1 = h^{-2}((2I + h^2 q_2) + (\delta_2 I + \beta_{2,c}))$$

$$\alpha_2 = h^{-2}(\gamma_2 I + \alpha_{2,c})$$

$$\beta_3 = \frac{a_3}{\alpha_2} = -\frac{I}{\gamma_2(I + \frac{\alpha_{2,c}}{\gamma_2})} = \delta_3 I + \beta_{3,c}$$

and finally

(19) $\alpha_j = h^{-2}(\gamma_j I + \alpha_{j,c}), \qquad \beta_j = \delta_j I + \beta j, c \qquad j = 3, ..n.$

where $\gamma_j, j = 1, ...n$ and $\delta_j, j = 2, ...n$, are numerical constants and $\alpha_{j,c}, j = 1, ...n$, $\beta_{j,c}, j = 2, ...n$, are operators which represent continuous functions.

It is clear that in this case $\gamma_1 = 2 \neq 0$ so $\delta_2 = -1/2 \neq 0$, and using the mathematical induction it can be proved easily that

$$\gamma_k = \frac{k+1}{k}, \qquad \delta_k = -\frac{k-1}{k}, \qquad k = 2, ...n.$$

This means that $\gamma_j \neq 0, j = 1, ...n$ and consequently from relation (19) it follows $\alpha_j \neq 0, j = 1, ...n$. As in linear algebra, in this case we can say that the matrix A is a regular matrix.

Let us suppose that in equation (6), $r = 0$ then the matrix A has the form

$$A = h^{-2} \begin{bmatrix} (2+q_1h^2)I & -I & 0 & .... & 0 & 0 \\ -I & (2+q_2h^2)I & -I & .... & 0 & 0 \\ 0 & -I & (2+q_3h^2)I & .... & 0 & 0 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ 0 & 0 & 0 & .... & -I & (2+q_nh^2)I \end{bmatrix}.$$

So, we can write

$$A = A_1 \cdot I$$

where the matrix A_1 has for elements numerical constants, i.e., it can be written as

$$(20)\quad A_1 = h^{-2} \begin{bmatrix} 2+q_1h^2 & -1 & 0 & .... & 0 & 0 \\ -1 & 2+q_2h^2 & -1 & .... & 0 & 0 \\ 0 & -1 & 2+q_3h^2 & .... & 0 & 0 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ 0 & 0 & 0 & .... & -1 & 2+q_nh^2 \end{bmatrix}.$$

From relations (15) it follows that d is a vector operator whose elements are representing continuous functions; using relation (18), we can prove the following

Corollary 1. *The numerical solutions of the differential equation (6) with conditions (5), with $p \geq m$ denoted by $v_k, k = 1, \dots n$, obtained as the solutions of the algebraic system given by relation (16) are operators from $\mathcal{F}$ representing continuous functions.*

3. The error of approximation

Let us denote by $\mathcal{F}_{cr}$ a subset of $\mathcal{F}_c$ consisting of continuous real valued functions. If $a = \{a(t)\}$ and $b = \{b(t)\}$ are from $\mathcal{F}_{cr}$, we can say that the operator a is greater or equal than the operator b, denoted by $a \geq b$, if $a(t) \geq b(t)$ for each $t \geq 0$ (see [2],p. 237).

Analogously, we shall say that $a(\lambda) \geq b(\lambda)$, if $a(\lambda)$ and $b(\lambda)$ are operational functions representing continuous real valued functions of two variables , $a(\lambda) = \{a(\lambda, t)\}$, $b(\lambda) = \{b(\lambda, t)\}$, such that $a(\lambda, t) \geq b(\lambda, t)$, for $t \geq 0, \lambda \in [0, 1]$.

The absolute value of an operator a from $\mathcal{F}_{cr}$, $a = \{a(t)\}$, denoted by $\mid a \mid$, is the operator $\mid a \mid = \{\mid a(t) \mid\}$. Also, we put $\mid a(\lambda) \mid = \{\mid a(\lambda, t) \mid\}$.

If the operators a and b are from $\mathcal{F}_{cr}$, then

$$\mid a + b \mid \leq \mid a \mid + \mid b \mid,$$

$$\mid ab \mid = \int_0^t a(\tau) b(t - \tau) d\tau \mid \leq \mid a \mid \mid b \mid,$$

and

$$\mid a \mid \leq_T A(T) l, \quad A(T) = \max_{t \in [0,T]}, \{\mid a(t) \mid\}.$$

If the operators a and b are not from $\mathcal{F}_{cr}$, but we can write them in a form $a = \alpha_1 I + a_1$ and $b = \beta_1 I + b_1$, where, now, a_1 and b_1 are operators from $\mathcal{F}_{c_r}$, then it holds

$$\mid l(\alpha_1 I + a_1) \mid \leq \mid \alpha_1 \mid l + \mid l a_1 \mid \leq \mid \alpha_1 \mid l + l \mid a_1 \mid .$$

Definition 1 *Let* **a** *be a vector operator given in the form* $\mathbf{a} = [a_1, a_2, \dots, a_n]^t$, *where $a_1, a_2, \dots, a_n$ are operators from $\mathcal{F}_{cr}$, and the superindex t stands for "transposed". Then we put*

$$\| a \|_\infty := \max_{1 \leq i \leq n} \mid a_i \mid .$$

In this paper we shall give the error of approximation for the the case when $r = 0$ in the equation (6). In that case we can consider matrix A_1 given by relation (20) instead of matrix A. For the matrix

$$A_0 = \begin{bmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ 0 & 0 & 0 & \dots & -1 & 2 \end{bmatrix}. \tag{21}$$

it holds $A_0 \leq A_1$ for $q(\lambda) \geq 0$ (meaning that each element of the matrix A_1 is greater or equal then corresponding element of the matrix A_0). The matrix A_0 is numerical matrix satisfying the following

$$r_{j,i} = \frac{i(n+1-j)h^2}{n+1}, \quad i \leq j, \qquad r_{i,j} = r_{j,i}.$$

In that case it is known (see [3]) that it holds

$$\| (A_0)^{-1} \|_\infty \leq \frac{h^2}{2} \begin{cases} (m+1)^2 & if \quad n = 2m+1 \\ m(m+1) & if \quad n = 2m \end{cases}$$

The infinite norm of a numerical matrix is defined in the usual way.

Since $h = \frac{1}{n+1}$, and $m + 1 = \frac{n+1}{2}$ for $n = 2m+1$ and $m(m+1) = \frac{n}{2}(\frac{n}{2}+1) < \frac{(n+1)^2}{2}$, it follows for numerical matrices A_0

$$\| A_0^{-1} \|_\infty \leq \frac{1}{8}.$$

Now, we can prove the following

Theorem 2. *Assume u is the exact solution of the system $Au = d + \tau(u)$, representing continuous functions and v its approximate solution, obtained as the solution of the matrix equation $Av = d$ (A, d and $\tau(u)$ are given by relations (20) and (16) respectively) and put in equation (6) $r = 0$. Then the error of approximation can be estimated as*

$$\| u - v \|_\infty \leq_T \frac{1}{96} M_4(T), \tag{22}$$

where

$$M_4(T) = \max_{t\in[0,T]} \max_{\lambda\in[0,1]} | u^{(4)}(\lambda, t) | . \tag{23}$$

Let us remark that in this case we have to estimate vector $u - v$, whose elements are Mikusiński operators $u_i - v_i$, $i = 1, 2, ..., n$. Therefore we shall estimate $\| u - v \|_\infty$ in the sense of the infinite norm.

Proof. From the relations

$$A(u - v) = \tau(u),$$

where A is given in (14) and $\tau(u)$ is given by (15) it follows

$$(u - v) = (A_1^{-1})\tau(u).$$

For numerical matrices A_0, A_1 it holds $A_1 \geq A_0$, for $q(\lambda) \geq 0$ and

$$A_0^{-1} \geq A_1^{-1}.$$

Since the elements of $\tau(u)$ represent the continuous functions, we can estimate $\| \tau(u) \|_\infty$ as (the infinite norm)

$$\| \tau(u) \|_\infty \leq \frac{h^2}{12} \max_{t\in[0,T]} \max_{\lambda\in[0,T]} u^{(4)}(\lambda, t) \leq \frac{h^2}{12} M_4.$$

From previous relations we can obtain the estimations (22).

References

[1] Mikusiński, J., Operational Calculus, Vol. I, Pergamon Press, Warszawa, 1983.

[2] Mikusiński, J., Boehme, T., Operational Calculus, Vol. II, Pergamon Press, Warszawa, 1987.

[3] Stojaković, Z., Herceg, D., Numeričke metode linearne algebre, Gradj. knjiga, Beograd, 1982 (in Serbian).

REZIME

DIFERENCNA ŠEMA U POLJU OPERATORA MIKUSINSKOG

Posmatrana je linearna diferencijalna jednačina reda 2 sa promenljivim koeficijentima u polju operatora Mikusinskog. Koristeći diskretni analogon posmatranog problema konstruisano je približno rešenje u svakoj tački $\lambda_i \in [0,1], i = 1, 2, ...n$. Odredjena je i greška aproksimacije.

Received by the editors December 4, 1990