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DEGREE OF L_1 APPROXIMATION TO INTEGRABLE FUNCTIONS BY INTEGRATED MEYER-KÖNIG AND ZELLER OPERATORS

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Abstract

The aim of the paper is to find an estimate for the degree of L_1 aproximation to integrable functions by integrated Meyer-König and Zeller operators.

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1. Introduction and results

It is well known that the n-th operator M_n, $n \in N$, of Meyer–König and Zeller is associated with a bounded function $f : I = [0,1] \to \Re$ the so-called n-th Bernstein power series.

$$M_n(f,x) = \sum_{k=0}^{\infty} m_{n,k}(x) f(\frac{k}{k+n}), \tag{1}$$

where

$$m_{n,k}(x) = \binom{k+n}{k} (1-x)^{n+1} x^k$$

converging for $0 \leq x < 1$. W. Meyer–König and Zeller [4] proved that the sequence $(M_n)_{n\in N}$ gives a linear approximation method on the on the normed space $(C(I), \|\ \|_\infty)$ (with $\|\ \|_\infty$ the usual sup-norm on I), i.e. $\lim_{n\to\infty}\|f - M_n f\|_\infty = 0$ for all $f \in C(I)$. Its degree of approximation can be estimated by [2].

$$\|f - M_n f\|_\infty \leq \frac{31}{27} w_{1,\infty}(f, \frac{1}{\sqrt{n}})\ \ (n \in N),$$

where $w_{1,\infty}(f,\ )$ is the ordinary modulus of continuity f with respect to the sup-norm.

A small modification of Meyer-König and Zeller operators due to M. W. Müller makes it possible to approximate Lebesque integrable functions in the L_1 norm by the integrated Meyer-König and Zeller operators

$$\hat{M}_n(f,x) = \sum_{k=0}^{\infty} m_{n,k}(x) \int_{\frac{k}{k+n}}^{\frac{k+1}{k+n+1}} f(t)dt,$$

where

$$\hat{m}_{n,k}(x) = (n+1) \binom{k+n+1}{k} (1-x)^n x^k.$$

The L_1 analogue of Meyer-König and Zeller's result was established by M. W. Müller [6] who has proved that for every Lebesque integrable function f on [0,1],

$$\int_0^1 |M_n(f,x) - f(x)|dx \to 0\ \ (n \to \infty).$$

As far as estimates of the degree of approximation to Lebesque integrable functions by the operators $\hat{M}_n(f)$ in the L_1 norm are concerned, very litlle is known. A results which gives the degree of approximation of f by some Bernstein type operators for a very special class of Lebesque integrable functions f is due to D. Leviatan [3]. Leviatan's result may be stated in our notation as follows:

If f is a Lebesque integrable function on [0,1], of bounded variation on every closed subinterval of (0,1), then

$$\int_0^1 |\hat{M}_n(f,x) - f(x)|dx \leq (2/e)^{1/2} J(f) n^{-1/2},$$

where

$$J(f) = \int_0^1 \sqrt{x}(1-x)|df(x)|.$$

This result is useful when $J(f) < \infty$.

In this paper we shall show that

$$\int_0^1 \sqrt{x}(1-x)|\hat{M}_n(f,x) - f(x)|dx$$

can be estimated in terms of the L_1 modulus of continuity

$$w_{L_1}(f,\delta) = \sup\{\int_0^1 |f(x+t) - f(x)|dx : |t| \le \delta\}.$$

We assume here and in the rest of paper that the function f is extended to $(-\infty, \infty)$ by periodicity with period 1 (its value at the integers is immaterial). The L_1 norm with weight function $w(x) = \sqrt{x}(1-x)$ seems to be a more convenient norm than the usual L_1 norm for the study of approximation properties of integrated Meyer-König and Zeller operators.

Our result may be stated as follows.

Theorem 1. *Let f be a Lebesque integrable function on [0,1]. Then, for $n \ge 2$,*

$$\int_0^1 \sqrt{x}(1-x)|\hat{M}_n(f,x) - f(x)|dx \le \frac{2\pi^2}{3} w_{L_1}(f, n^{-\frac{1}{2}})$$

where

$$w_{L_1}(f,\delta) = \sup\{\int_0^1 |f(x+t) - f(x)dx : |t| \le \delta\}.$$

The proof will be tailored specially for the case of the L_1 norm and follows ideas in a paper by Bojanić and Shisha [1].

2. Lemmas

The proof of our theorem is based on two lemmas.

Lemma 1. *If f is a Lebesque integrable function on [0,1], then for $n \ge 2$ $(n \in N)$ and $x, t \in [0,1]$, we have*

$$x(1-x)^2(\hat{M}_{n-1}(f,x) - f(x)) \le$$

$$\le \sum_{k=0}^{\infty} n m_{n,k}(x)(\frac{k}{n+k} - x) \int_0^{\frac{k}{n+k-1} - x} (f(x+t) - f(x))dt.$$

Proof. We have

$$\hat{M}_{n-1}(f,x)=\int_0^1 K_n(x,t)f(t)dt,$$

where

$$K_n(x,t)=\sum_{k=0}^{\infty}\hat{m}_{n-1,k}(x)\chi_{[\frac{k}{n+k-1},\frac{k+1}{n+k}]}(t),$$

$\chi_{[\frac{k}{n+k-1},\frac{k+1}{n+k}]}(t)$, being the characteristic function of $[\frac{k}{n+k-1},\frac{k+1}{n+k}]$. By partial summation we find for $m\in N$, $n\geq 2$ and $x,t\in[0,1]$ that

$$\sum_{k=0}^{m}\hat{m}_{n-1,k}(x)\chi_{[\frac{k}{n+k-1},\frac{k+1}{n+k}]}(t)=$$

$$=\sum_{k=0}^{m}(\hat{m}_{n-1,k-1}(x)-\hat{m}_{n-1,k}(x))\chi_{[0,\frac{k}{k+n-1}]}(t)+\hat{m}_{n-1,m}(x)\chi_{[0,\frac{m+1}{n+k-1}]}(t).$$

Since
$\hat{m}_{n-1,k-1}(x)-\hat{m}_{n-1,k}(x)=$

$$\begin{aligned}
&= n\begin{pmatrix} k+n-1 \\ k-1 \end{pmatrix} x^{k-1}(1-x)^{n-1}-n\begin{pmatrix} k+n \\ k \end{pmatrix} x^{k}(1-x)^{n-1}\\
&= n\left[\begin{pmatrix} n+k-1 \\ k-1 \end{pmatrix}-\begin{pmatrix} n+k \\ k \end{pmatrix}x\right]x^{k-1}(1-x)^{n}\\
&= n\begin{pmatrix} n+k \\ k \end{pmatrix}\left[\frac{k}{k+n}-x\right]x^{k-1}(1-x)^{n-1},
\end{aligned}$$

and

$$\lim_{m\to\infty}\hat{m}_{n-1,m}(x)\to 0,$$

we have

$$x(1-x)^2(\hat{m}_{n-1,k-1}(x)-\hat{m}_{n-1,k}(x))=$$

$$n(\begin{matrix} n+k \\ k \end{matrix})x^k(1-x)^{n+1}(\frac{k}{n+k}-x)=nm_{n,k}(x)(\frac{k}{n+k}-x).$$

Now, it follows that

$$x(1-x)^2K_n(x,t)=\sum_{k=0}^{\infty}nm_{n,k}(x)[\frac{k}{n+k}-x]\chi_{[0,\frac{k}{n+k-1}]}(t).$$

Hence

$$x(1-x)^2\hat{M}_{n-1}(f,x) = \sum_{k=0}^{\infty} nm_{n,k}(x)(\frac{k}{n+k}-x)\int_0^{\frac{k}{n+k-1}} f(t)dt$$

$$= \sum_{k=0}^{\infty} nm_{n,k}(x)(\frac{k}{n+k}-x)\int_x^{\frac{k}{n+k-1}} f(t)dt.$$

Thus, the proof of the lemma is complete, since

$$\begin{aligned}\int_0^{\frac{k}{n+k-1}} f(t)dt &= \int_0^x f(t)dt + \int_x^{\frac{k}{n+k-1}} f(t)dt \\ &= \int_0^x f(t)dt + \int_0^{\frac{k}{n+k-1}-x} f(x+t)dt\end{aligned}$$

and

$$\sum_{k=0}^{\infty}(\frac{k}{n+k}-x)^2 m_{n,k}(x) = \frac{x(1-x)^2}{n} + \frac{x(1-x)^2(2x-1)}{n^2} + O(n^{-3}).$$

Our second lemma is a more precise version of the known equalities (see [8], 431–435 and [7]).

Lemma 2. *For $n \geq 2$ and $x \in [0,1]$ we have*

$$\sum_{k=0}^{\infty} |\frac{k}{n+k}-x|^5 m_{n,k}(x) \leq x(1-x)^2/n^{\frac{5}{2}}.$$

Proof. We have

$$\sum_{k=0}^{\infty} |\frac{k}{n+k}-x|^5 m_{n,k}(x) \leq$$

$$(\sum_{k=0}^{\infty}(\frac{k}{n+k}-x)^4 m_{n,k}(x))^{\frac{1}{2}}(\sum_{k=0}^{\infty}(\frac{k}{n+k}-x)^6 m_{n,k}(x))^{\frac{1}{2}},$$

and the result follows, since

$$\begin{aligned}\sum_{k=0}^{\infty}(\frac{k}{n+k}-x)^4 m_{n,k}(x) &= \sum_{k=0}^{\infty}(\frac{k}{n+k})^4 m_{n,k}(x) - \sum_{k=0}^{\infty} 4x(\frac{k}{n+k})^3 \cdot \\ &\quad m_{n,k}(x) + \sum_{k=o}^{\infty} 6x^2(\frac{k}{n+k})^2 m_{n,k}(x) - 3x^4\end{aligned}$$

$$
\begin{aligned}
&= x\left[x^3+\frac{3x^2(1-x)^2}{n}+\frac{x(1-x)^2(1-2x+11x^2)}{n^2}\right.\\
&\quad+\frac{3x^3(1-x)^4}{n}+\frac{3x^3(1-x)^2}{n^2}-\frac{21x^3(1-x)^3}{n^2}\\
&\quad\left.+\frac{3x^2(1-x)^4}{n^2}+\frac{3x^2(1-x)^3}{n^2}\right]-4x\left[x^3\right.\\
&\quad\left.+\frac{3x^2(1-x)^2}{n}+\frac{x(1-x)^2(1-2x+11x^2)}{n^2}\right]\\
&\quad+6x^2\left[x^2+\frac{x(1-x)^2}{n}+\frac{x(1-x)^2(2x-1)}{n^2}\right]\\
&\quad-3x^4+O(n^{-3})\\
&= \frac{3x^2(1-x)^4}{n^2}+O(n^{-3})\leq\frac{x(1-x)^2}{n^2}+O(n^{-3})
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{k=0}^{\infty}\left(\frac{k}{n+k}\right)^6 m_{n,k}(x) &= \sum_{k=0}^{\infty}\left(\frac{k}{n+k}\right)^6 m_{n,k}(x)-\sum_{k=0}^{\infty}6x\left(\frac{k}{n+k}\right)^5 m_{n,k}(x)\\
&\quad+\sum_{k=o}^{\infty}15x^2\left(\frac{k}{n+k}\right)^4 m_{n,k}(x)-\sum_{k=o}^{\infty}20x^3\left(\frac{k}{n+k}\right)^3\cdot\\
&\quad m_{n,k}(x)+\sum_{k=o}^{\infty}15x^4\left(\frac{k}{n+k}\right)^2 m_{n,k}(x)-5x^6\\
&= \frac{5x^3(1-x)^6}{n^3}+O(n^{-4})\leq\frac{x(1-x)^2}{n^3}+O(n^{-4})
\end{aligned}
$$

for $x\in[0,1]$.

3. Proof of the theorem

Let $x\in(0,1)$. By lemma 1 we have

$$x(1-x)^2|\hat{M}_{n-1}(f,x)-f(x)|\leq$$

$$\leq\sum_{k=0}^{\infty}nm_{n,k}(x)\left|\frac{k}{n+k}-x\right|\left|\int_0^{\frac{k}{n+k-1}-x}(f(x+t)-f(x))dt\right|$$

$$\leq \sum_{k=0}^{\infty} nm_{n,k}(x)\left|\frac{k}{n+k}-x\right| \int_{-\left|\frac{k}{n+k-1}-x\right|}^{\left|\frac{k}{n+k-1}-x\right|} |(f(x+t)-f(x))dt|$$
$$\leq \sum_{r=0}^{[1/\delta]} I_{n,r}(x),$$

where $\delta \in (0,1)$ and

$$I_{n,r}(x) = \sum_{r\delta < \left|\frac{k}{n+k-1}-x\right| \leq (r+1)\delta} nm_{n,k}(x)\left|\frac{k}{n+k}-x\right| \cdot$$
$$\int_{-\left|\frac{k}{n+k-1}-x\right|}^{\left|\frac{k}{n+k-1}-x\right|} |f(x+t)-f(x)|dt.$$

Clearly,

$$I_{n,r}(x) \leq S_r(n,\delta;x) \int_{-(r+1)\delta}^{(r+1)\delta} |f(x+t)-f(x)|dt,$$

where

$$S_r(n,\delta;x) = \sum_{r\delta < \left|\frac{k}{n+k-1}-x\right| \leq (r+1)\delta} nm_{n,k}(x)\left|\frac{k}{n+k}-x\right|.$$

Hence, it follows that

$$x(1-x)^2|\hat{M}_{n-1}(f,x)-f(x)| \leq$$

$$\leq \sum_{r=0}^{[1/\delta]} S_r(n,\delta;x) \int_{-(r+1)\delta}^{(r+1)\delta} |f(x+t)-f(x)|dt. \tag{2}$$

Next we shall estimate the coefficient $S_r(n,\delta;x)$ for $r=0$ and $1 \leq r \leq [1/\delta]$. We have first

$$\begin{aligned} S_0(n,\delta;x) &= \sum_{\left|\frac{k}{n+k}-x\right| \leq \delta} nm_{n,k}(x)\left|\frac{k}{n+k}-x\right| \\ &\leq \sum_{k=0}^{\infty} nm_{n,k}(x)\left|\frac{k}{n+k-1}-x\right| \\ &\leq n^{1/2}\sqrt{x}(1-x)+O(n^{-2}). \end{aligned} \tag{3}$$

Next, for $1 \le r \le [1/\delta]$, we have, by Lemma 2,

$$\begin{aligned} S_r(n,\delta;x) &\le n(r+1)^{-4}\delta^{-4} \sum_{r\delta<|\frac{k}{n+k-1}-x|\le(r+1)\delta} |\frac{k}{n+k}-x|^5 m_{n,k}(x) \\ &\le n(r+1)^{-4}\delta^{-4}\sum_{k=0}^{\infty}|\frac{k}{n+k}-x|^5 m_{n,k}(x) \\ &\le n^{-3/2}x(1-x)^2(r+1)^{-4}\delta^{-4}+O(n^{-3}). \qquad (4) \end{aligned}$$

From (2), (3) and(4) it follows that

$$\sqrt{x}(1-x)|\hat{M}_{n-1}(f,x)-f(x)| \le n^{1/2}\int_{-\delta}^{\delta}|f(x+t)-f(x)|dt+$$

$$+\frac{1}{2}n^{-3/2}\delta^{-4}\sum_{r=1}^{[1/\delta]}(r+1)^{-4}\int_{-(r+1)\delta}^{(r+1)\delta}|f(x+t)-f(x)|dt.$$

Integrating this inequality and taking into account that

$$\int_{-(r+1)\delta}^{(r+1)\delta}\left(\int_0^1|f(x+t)-f(x)|dx\right)dt \le 2(r+1)\delta w_{L_1}(f,(r+1)\delta),$$

we find that

$$\int_0^1\sqrt{x}(1-x)|\hat{M}_{n-1}(f,x)-f(x)|dx$$

$$\le 2n^{1/2}\delta w_{L_1}(f,\delta)+n^{-3/2}\delta^{-3}\sum_{r=1}^{[1/\delta]}(r+1)^{-3}w_{L_1}(f,(r+1)\delta).$$

Choosing here $\delta = n^{-1/2}$, we find that

$$\int_0^1\sqrt{x}(1-x)|\hat{M}_{n-1}(f,x)-f(x)|dx$$

$$\le 2w_{L_1}(f,n^{-1/2})+\sum_{r=0}^{[1/n^{-1/2}]}(r+1)^{-3}w_{L_1}(f,(r+1)/n^{1/2})$$

$$\le 2\sum_{k=1}^{[1/n^{-1/2}]+1}k^{-3}w_{L_1}(f,k/n^{1/2}).$$

Since the L_1 modulus of continuity is a subadditive function, we have, for every $0 < h_1 \leq h_2$,

$$\frac{2w_{L_1}(f,h_1)}{h_1} \geq \frac{w_{L_1}(f,h_2)}{h_2}$$

(see [9], p. 112). In particular we have, for $k \geq 1$,

$$w_{L_1}(f, k/n^{1/2}) \leq 2kw_{L_1}(f, n^{-1/2}).$$

Hence,

$$\int_0^1 \sqrt{x}(1-x)|\hat{M}_{n-1}(f,x) - f(x)|dx$$

$$\leq 4w_{L_1}(f, n^{-1/2}) \sum_{k=1}^{\infty} k^{-2} \leq \frac{2\pi^2}{3} w_{L_1}(f, n^{-1/2})$$

and the theorem is proved.

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REZIME

STEPEN L_1 APROKSIMACIJE INTEGRABILNIH FUNKCIJA POMOĆU MEYER - KÓNIG I ZELLER INTEGRALNIH OPERATORA

U radu je odredjena ocena za stepen L_1 aproksimacije integrabilnih funkcija pomoću MEYER - KÓNIG i ZELLER integralnih operatora

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