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A THEOREM ON BEST APPROXIMATIONS AND APPLICATIONS

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Abstract

A theorem on best approximations in pseudoconvex metric spaces is proved. As corollaries some results of Rassias [7] and Kassay [4] are obtained and two propositions on the fixed points are proved.

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1.

It is well known that KKM theory is very useful in the fixed point theory and the approximation theory [5].

In this paper we shall use a generalization of the well known lemma of Knaster, Kuratowski and Mazurkiewicz proved by Ch. Horvath [3] in order to obtain a theorem on best approximations and some fixed point results. The following definition is introduced in [3].

Definition 1. *Let X be a topological space and $h : X \times X \times [0,1] \to X$ so that:*

(i) $h(x,y,0) = y$, $h(x,y,1) = x$, *for every* $(x,y) \in X \times X$.

(ii) For every finite subset A of X, the restriction of h on $\mathrm{co}_h(A) \times \mathrm{co}_h(A) \times [0,1]$ is continuous, where $\mathrm{co}_h(A)$ is the convex hull of A with respect to h.

Then h is a pseudoconvex structure on X and (X,h) is a pseudoconvex space.

Let (X,h) be a pseudoconvex space and $R : X \to 2^X$. The mapping R is said to be an element of $KKM_h(X)$ [2] if for every finite subset A of X, $\mathrm{co}_h(A) \subseteq R(A)$. Ch. Horvath proved in [3] the following theorem.

Theorem A. *Let (X,d,h) be a complete pseudoconvex metric space and $R \in KKM_h(X)$ such that $R(x)$ is closed for every $x \in X$. If for every $\epsilon > 0$ there exists a finite set A such that $\alpha(\bigcap_{x\in A} R(x)) < \epsilon$, where α is the Kuratowski measure of noncompactness, then $\bigcap_{x\in X} R(x)$ is nonempty and compact.*

Definition 2. *Let (X,d,h) be a pseudoconvex metric space, M a nonempty, convex subset of X and $g : M \to X$. The mapping g is said to be $h-$ almost affine if for every finite set $\{x_1, x_2, ..., x_n\} \subset M$ and every $z \in X$ the following implication holds:*

$$y \in \mathrm{co}_h\{x_1, x_2, ..., x_n\} \Rightarrow d(g(y), z) \leq \max_{1\leq i\leq n} d(g(x_i), z).$$

If $(X, \|\cdot\|)$ is a normed space, M a convex subset of X and $g : M \to X$ is an almost affine mapping in the sense of Prolla [6] then g is $h-$ almost affine for $h(x,y,\lambda) = \lambda x + (1-\lambda)y$ $(x,y \in X; \lambda \in [0,1])$.

In 1970. Takahashi introduced the notion of a metric space with a convex structure.

Definition 3. *Let (X,d) be a metric space and $W : X \times X \times [0,1] \to X$. The mapping W is a convex structure on X if for all $x, y \in X$ and $\lambda \in [0,1]$*

$$d(u, W(x,y,\lambda)) \leq \lambda d(u,x) + (1-\lambda)d(u,y)$$

for every $u \in X$. Then (X,d,W) is a convex metric space.

If W is continuous or $\mathrm{co}_W(A)$ is compact for every finite $A \subset X$ then (X,d,W) is a pseudoconvex metric space. If (X,d,W) is strongly convex metric space in the sense of Talman $\mathrm{co}_W(A)$ is compact for every finite A [10].

2.

Now, we can prove the following theorem on best approximations.

Theorem 1. *Let (X,d,h) be a pseudoconvex metric space, $\emptyset \neq M$ a convex and complete subset of X, g a continuous $h-$almost affine mapping of M onto M and $f : M \to X$ a continuous mapping such that*

$$\inf_{x\in M} \alpha[\{y; y \in M, d(g(y), f(y)) \leq d(g(x), f(y))\}] = 0. \tag{1}$$

Then there exists $y_0 \in M$ such that

$$d(g(y_0), f(y_0)) = d(f(y_0), M). \tag{2}$$

Proof. It is easy to verify that the mapping $R : M \to 2_c^M$ satisfies all the conditions of Theorem A, where:

$$R(x) = \{y; y \in M, d(g(y), f(y)) \leq d(g(x), f(y))\},$$

for every $x \in M$.

We shall prove that $R \in KKM_h(M)$. Let $A = \{x_1, x_2, ..., x_n\} \subset M$. Suppose that $y \in co_h(A)$. If $y \notin R(A)$ then $y \notin R(x_i), i \in \{1, 2, ..., n\}$ which means that

$$d(g(y), f(y)) > d(g(x_i), f(y)), i \in \{1, 2, ..., n\}.$$

But, g is $h-$almost affine which implies that

$$d(g(y), f(y)) \leq \max_{1\leq i\leq n} d(g(x_i), f(y)).$$

Hence $y \notin R(A)$ leads to a contradicition.

Remark: Theorem 1 from [1] is proved in [2] (see Remark after Theorem 2 in [2]) and it is a Corollary of Theorem 1.

Corollary 1. *Let (X,d,W) be a convex metric space such that W is continuous or $co_W(A)$ is compact for every finite $A \subset X$. Let $\emptyset \neq M$ be a convex and complete subset of X, g a continuous W- almost affine mapping of M onto M and $f : M \to X$ a continuous mapping such that (1) holds. Then there exists $y_0 \in M$ such that (2) holds.*

Corollary 2. *Let $(X,d,W),M$ and f be as in Corollary 1 and (1) holds for $g(x)=x$, $x\in M$. Then there exists $y_0\in M$ such that*

$$d(y_0,f(y_0))=d(f(y_0),M).$$

Proof. It remains to be proved that $g(x)=x$, $x\in M$ is $W-$ almost affine. It is known that for every $A\subset X$

$$\mathrm{co}A=\bigcup_{n\in\mathbf{N}}\tilde{W}^n(A),\tilde{W}^n(A)=W(\tilde{W}^{n-1}(A)),\ n\geq 2$$

$$\tilde{W}^1(A)=\{W(x,y;\lambda);\lambda\in[0,1],x,y\in A\}.$$

This implies that for every $y\in \mathrm{co}_W\{x_1,x_2,...,x_n\},\{x_1,x_2,...,x_n\}\subset M$ there exist $\lambda_i\geq 0$, $i\in\{1,2,...,n\},\sum_{i=1}^n\lambda_i=1$, such that for every $z\in X$:

$$d(y,z)\leq\sum_{i=1}^n\lambda_i d(x_i,z)\leq\max_{1\leq i\leq n}d(x_i,z).$$

Corollary 3. [4] *Let (X,d,W) be a compact, convex metric space with continuous convex structure W. If $f:X\to X$ is a continuous mapping then $Fix(f)\neq\emptyset$.*

The last Corollary is proved by Rassias in [7].

Corollary 4. *Let (X,d) be a metric linear topological space such that all balls are convex and A be a nonempty, compact and convex subset of X. If $f:A\to X$ is a continuous mapping then there exists at least one point $y_0\in A$ such that $d(y_0,f(y_0))=d(f(y_0),A)$.*

Proof. We shall prove that $g(x)=x$, $x\in X$ is an $h-$ almost affine mapping, where $h(x,y,\lambda)=\lambda x+(1-\lambda)y$ for every $x,y\in X$ and $\lambda\in[0,1]$. If $y\in \mathrm{co}_h\{x_1,x_2,...,x_n\},\{x_1,x_2,...,x_n\}\subset A$ then

$$y=\sum_{i=1}^n\lambda_i x_i,\lambda_i\geq 0,i\in\{1,2,...,n\},\sum_{i=1}^n\lambda_i=1$$

and let for every $z\in X,\max_{1\leq i\leq n}d(x_i,z)=r_z$. Hence, $x_i\in B(z,r_z),=\{x;d(x,z)\leq r_z\}$ for every $i\in\{1,2,...,n\}$ which implies that $\sum_{i=1}^n\lambda_i x_i\in B(z,r_z)$. This means that $d(y,z)\leq r_z$ and so g is an $h-$ almost affine mapping.

3.

If (X,d) is a strongly convex metric space, by using Corollary 3 we can prove two propositions on fixed points.

Definition 4. [10] *Let (X,d) be a metric space and*

$$P = \{(t_1,t_2,t_3);(t_1,t_2,t_3) \in [0,1] \times [0,1] \times [0,1], t_1 + t_2 + t_3 = 1\}.$$

A strongly convex structure (SCS) on X is a continuous function $K : X \times X \times X \times P \to X$ with the property that for each $(x_1,x_2,x_3,t_1,t_2,t_3) \in X \times X \times X \times P, K(x_1,x_2,x_3,t_1,t_2,t_3)$ is the unique point of X which satisfies

$$d(y, K(x_1,x_2,x_3,t_1,t_2,t_3)) \leq \sum_{k=1}^{3} t_k d(y,x_k).$$

for every $y \in X$.

If (X,d) is a strongly convex metric space and K its SCS then $W_K : X \times X \times [0,1] \to X$, defined by $W_K(x_1,x_2,t) = K(x_1,x_2,x_1,t,1-t,0)$ is a Takahashi convex structure.

If $S \subset X$ is convex and $r > 0$ let $S_r = \{x \in X; d(x,S) < r\}$. A subset S of X, where (X,d) is a strongly convex metric space, is stable if S_r is convex for every $r > 0$. The SCS on X is stable if the set $\{W(x,y,t);\ t \in [0,1]\}$ is stable for every pair $x,y \in X$. In [9] it is proved that a SCS on X is stable if and only if every convex subset of X is stable.

If, for example, W_K satisfies the condition that for every $x_1,x_2,y_1,y_2 \in X$ and $\lambda \in [0,1]$

$$(3) \qquad d(W_K(x_1,x_2;\lambda), W_K(y_1,y_2,\lambda)) \leq \lambda d(x_1,y_1) + (1-\lambda)d(x_2,y_2)$$

then X is stable. Let $S \subset X$ be convex and $r > 0$. Indeed from $x_i \in S_r$, $i \in \{1,2\}$ it follows that there exist $y_i \in S$, $i \in \{1,2\}$ such that $d(x_1,y_1) < r$ and $d(x_2,y_2) < r$. Then from the convexity of S it follows that $W(y_1,y_2,\lambda) \in S$ for every $\lambda \in [0,1]$. So for every $\lambda \in [0,1]$

$$d(W_K(x_1,x_2;\lambda), W_K(y_1,y_2;\lambda)) \leq \lambda d(x_1,y_1) + (1-\lambda)d(x_2,y_2) < r,$$

hence $W_K(x_1,x_2,\lambda) \in S_r$ which means that S_r is convex for every $r > 0$.

Proposition 1. *Let (X,d,W) be a convex metric space with continuous convex structure W and A a nonempty convex and complete subset of X such that for every compact $M \subset A$ is $\mathrm{co}M$ precompact. Let $f : A \to A$ be a compact mapping. Then $Fix(f) \neq \emptyset$.*

Proof. Since $f(A)$ is compact it follows that $\mathrm{co}f(A)$ is precompact and from the completeness of A it follows that $\overline{\mathrm{co}}f(A)$ is a compact subset of A. Then f is a continuous mapping from $\overline{\mathrm{co}}f(A)$ into $\overline{\mathrm{co}}f(A)$ and applying Corollary 3 we obtain that $Fix(f) \neq \emptyset$.

Remark: If (X,d) is a strongly convex metric space with a stable SCS then [9] the following implication holds: M is precompact $\Rightarrow$ $\mathrm{co}M$ is precompact.

Proposition 2. *Let (X,d) be a strongly convex metric space with a SCS K such that W_K satisfies (3). Let A be a convex and complete subset of X and $T,S : A \to X$ continuous mappings such that the following conditions are satisfied:*

(i) $d(Tx,Ty) \leq qd(x,y)$ *, for every* $x,y \in A$*, where* $q \in [0,1]$.

(ii) $\overline{S(A)}$ *is compact.*

(iii) For every $\lambda \in [0,1]$, $W_K(Tx,Sy,\lambda) \in A$.

If $0 \leq q \leq 1$ *and* $\lambda \in [0,1)$ *or* $0 \leq q < 1$ *and* $\lambda \in [0,1]$ *then* $Fix(\phi_\lambda) \neq \emptyset$, *where*

$$\phi_\lambda x = W_K(Tx,Sx,\lambda), \quad x \in A.$$

Proof. Let, for every $y \in S(A)$:

$$G_y x = W_K(Tx,y;\lambda), \quad x \in A.$$

Then $G_y : A \to A$ and

$$d(G_y x_1, G_y x_2) = d(W_K(Tx_1,y;\lambda), W_K(Tx_2,y;\lambda))$$

$$\leq \lambda d(Tx_1,Tx_2) \leq \lambda q d(x_1,x_2)$$

for every $x_1,x_2 \in A$. Since $\lambda q \in [0,1)$, for every $y \in S(A)$ there exists $Ry \in A$ such that $G_y(Ry) = W_K(TRy,y;\lambda) = Ry$ and the mapping $y \mapsto Ry (y \in S(A)$ is continuous on $S(A)$ since $y \mapsto G_y x$ is continuous. Let $fx = RSx$, $x \in A$.

Then f is continuous and from the compactness of $\overline{S(A)}, \overline{f(A)} = \overline{R(S(A))}$ is compact as well. From Proposition 1 it follows that $Fix(f) \neq \emptyset$. Then $y_0 \in Fix(f)$ implies that

$$y_0 = RSy_0 = W_K(TRSy_0, Sy_0; \lambda) = W_K(Ty_0, Sy_0; \lambda) = \phi_\lambda y_0.$$

In the next Proposition we shall use the notation :

$$N_\varepsilon(M) = \{x; x \in X, d(x, M) \leq \varepsilon\}$$

where $\varepsilon > 0, (X, d)$ is a metric space and $M \subset X$.

Proposition *Let (X, d) be a strongly convex metric space, M a nonempty, convex and complete subset of $X, \varepsilon > 0$ and $f : M \to N_\varepsilon(M)$ a continuous mapping such that $f(M)$ is bounded. Then*

$$\inf_{x \in K} d(x, f(x)) \leq \varepsilon + \alpha[f(M)].$$

Proof. Let $\delta > 0$ and $\{f(x_1), f(x_2), ..., f(x_n)\}$, be an $\alpha[f(M)] + \delta/2-$ net of $f(M)$, which means that

$$f(M) \subseteq \bigcup_{i=1}^{n} L(f(x_i), \alpha[f(M)] + \delta/2).$$

Since $f(x_i) \in N_\varepsilon(M)$ there exists $w_i \in M$ such that $d(w_i, f(x_i)) < \varepsilon + \delta/2, i \in \{1, 2, ..., n\}$. The set

$$H = co_{W_M}\{x_1, x_2, ..., x_n, w_1, w_2, ..., w_n\}$$

is convex and compact since X is a strongly convex metric space. Hence from Theorem 1 it follows the existence of an element $y \in H$ such that $d(y, f(y)) = d(f(y), H)$. On the other hand $d(f(y), H) < \varepsilon + \delta + \alpha[f(M)]$.

Indeed,for such an $y \in H$ there exists $x_i \in H$ such that

$$d(f(y), f(x_i)) < \alpha[f(M)] + \delta/2$$

and $w_i \in H$ such that $d(w_i, f(x_i)) < \varepsilon + \delta/2$.

This implies that

$$d(f(y), w_i) < \varepsilon + \alpha[f(M)] + \delta$$

and so $d(f(y), H) \leq \varepsilon + \alpha[f(M)]$.

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REZIME

TEOREMA O NAJBOLJOJ APROKSIMACIJI I PRIMENE

Dokazana je teorema o najboljoj aproksimaciji u pseudokonveksnim metričkim prostorima. Kao posledice dobijeni su neki rezultati Rassiasa [7] i Kassaya [4] i dokazana dva tvrdjenja o nepokretnoj tački.

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