Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 22, 2 (1992), 187-196

Review of Research
Faculty of Science
Mathematics Series

APPROXIMATE CONVEX HULLS–EFFICIENCY EVALUATIONS

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Abstract

The paper gives some efficiency evaluations of an approximate convex hull algorithm for finite grid point sets proposed by Stojmenović, and gives an algorithm for the determination of function f' which characterizes the efficiency of the proposed algorithm.

AMS Mathematics Subject Classification (1991): 68E99
Key words and phrases: conves hull, grid point set, extreme point, inner approximation, outer apporoximation.

1. Introduction

The convex hull CH(S) of a point set S is the smallest convex set that contains the set S.The problem of constructing the convex hull of a finite point set arises in many applications.But it is still possible that for some special applications (for example: a large number of small objects in a picture) certain approximations of the convex hull should be preferred to reduce the computation time, without any essential loss of efficiency.

We shall observe an approximate convex hull algorithm for a finite grid point set.

A grid point set G is a finite set of points $P = (x_1, x_2)$ in the regular orthogonal grid having integer coordinates: x_1, x_2. Klete gave an approximation algorithm for determining the convex hull of a finite grid point set.

For a direction $\alpha \in [0, 2\pi]$ and a grid point $P = (x_1, x_2)$ let $g(\alpha, P)$ be the straight line passing through point P in the direction $\alpha + \pi/2$. Points $Y = (y_1, y_2)$ of $g(\alpha, P)$ satisfy the equation:

$$Z_\alpha(y) : (y_2 - x_2) \sin\alpha + (y_1 - x_1) \cos\alpha = 0$$

A point $P \in G$ is called an extreme point in direction α of the grid point set $G, (P \in Ex_\alpha(G))$ iff $Z_\alpha(P) = 0$ and $Z_\alpha \leq 0$ for all $Y \in G$.

By $hp(\alpha, P)$ we denote the halfplane defined by $g(\alpha, P)$ and containing the set G.

Now, let $dir(n) = \{0, \frac{1}{n}2\pi, \frac{2}{n}2\pi, \ldots, \frac{n-1}{n}2\pi\}$ be the set of n-directions.

Let $H_n(G) = \bigcap_{\alpha \in dir(n)} hp(\alpha, G)$ be the outer approximation of $CH(G)$ called the n-hull of $G, (CH(G) \subset H_n(G))$.
Let $A_n(G) = CH(\bigcup_{\alpha \in dir(n)} Ex_\alpha(G))$ be the inner approximation of $CH(G)$, called the n-approximation $(A_n(G) \subset CH(G))$.
Let

$$xdiam(G) = \max\{|x_1 - y_1| \mid (x_1, x_2) \in G, (y_1, y_2) \in G\}$$

$$ydiam(G) = \max\{|x_2 - y_2| \mid (x_1, x_2) \in G, (y_1, y_2) \in G\}$$

$$diam(G) = \max\{xdiam(G), ydiam(G)\}$$

Klette defined functions $f(n), g(n)$ such that $f(n)$ is a maximal integer m, such that for all the grid points sets G, with $diam(G) \leq m$, the n-approximation $A_n(G)$ is always equal to $CH(G)$. $g(m)$ is the minimal integer n such that $f(n) \geq m$ for $m > 0$. (Note: $g(m)$ is the minimal number of necessary directions for which $CH(G) = A_n(G)$ whenever $diam(G) \leq m$).

Stojmenović gave a new approximate convex hull algorithm for a finite grid point set.

Different from Klette's algorithm Stojmenović proposes a new set of directions

$$dir'(n) = \{\arctan\frac{n}{i},\ \arctan\frac{i}{n},\ i \in \{-n \leq i \leq n\}\},$$

so, that the computation of the extreme point in direction α and $\alpha+\pi$ could be done at the same time, that is, point P is the extreme point in direction $\alpha \in dir'(n)$ of the grid point set $G, (P \in Ex'_\alpha(G))$ iff
$Z_\alpha(P) = 0$ and $Z_\alpha(Y) \leq 0$ for all $Y \in G$ or
$Z_\alpha(P)$ and $Z_\alpha(Y) \geq 0$ for all $Y \in G$.
This means:

$$P \in Ex'_\alpha(G) \Leftrightarrow P \in Ex_\alpha(G) \text{ or } P \in Ex_{\alpha+\pi}(G).$$

Other definitions are given analogously

$A'_n(G) = CH(\bigcup_{\alpha \in dir'(n)} Ex'_\alpha(G))$ -inner approximation

$H'_n(G) = \bigcap_{\alpha \in dir'(n)} hp(\alpha, G)$-outer approximation

$f'(n)$-is the maximal integer m, such that for all the grid point sets with $diam(G) \leq m$, the n-approximation $A'_n(G)$ is always equal to CH(G)

$g'(m)$-is the minimal integer n, such that: if $diam(G) \leq m$ then $A'_n(G) = CH(G)$.

Klette mentioned an open problem whether there is an n such that $H_n(G) = CH(G)$ for a given grid point set G. For the algorithm proposed by Stojmenović for each grid point set G there is an n such that $H'_n(G) = A'_n(G) = CH(G)$.

2. Behaviour of functions $f'(n)$ and $g'(n)$

In this section the behaviour of functions $f'(n), g'(n)$ will be evaluated.

Lemma 1.

$$g'(m) \geq \frac{m_*^2 - 2m_*}{2} \text{ where } m_* = [\frac{m-1}{2}].$$

Proof. Let $g'(m) = t$.
Let us observe grid point $sets G_a = \{A, B, C\}$ as showninig Fig. 1. for $a = m_*,\ m_* - 1,\ m_* - 2$. Because diam $(G_a) \leq m$ and $g'(m) = t$, it follows $CH(G_a) = A'_t(G_a)$ for $a = m_*,\ m_* - 1,\ m_* - 2$.

This means: integers k_1, k_2, k_3, such that:

$$\arctan\frac{k_1}{t} \in [\arctan\frac{1}{m_*+1}, \arctan\frac{1}{m_*}]$$

$$\arctan\frac{k_2}{t} \in [\arctan\frac{1}{m_*}, \arctan\frac{1}{m_*-1}]$$

$$\arctan\frac{k_3}{t} \in [\arctan\frac{1}{m_*-1}, \arctan\frac{1}{m_*-2}]$$

must exist. (Because $B \in Ex'_\alpha(G)$)

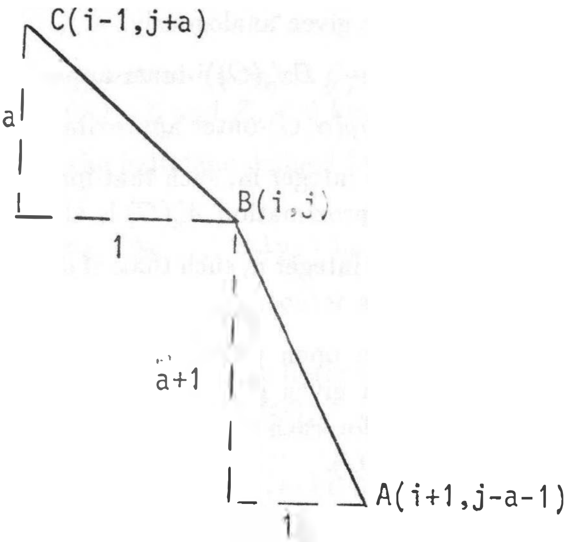

Fig. 1

From (I) we have:

$$\frac{k_1}{t} \in [\frac{1}{m_*+1}, \frac{1}{m_*}]$$

$$\frac{k_2}{t} \in [\frac{1}{m_*}, \frac{1}{m_*-1}]$$

$$\frac{k_3}{t} \in [\frac{1}{m_*-1}, \frac{1}{m_*-2}]$$

1 - case: If $\frac{k_1}{t} \neq \frac{1}{m_*}$ then (since $k_1 \neq k_2$)

$$\frac{1}{t} < \frac{1}{m_*-1} - \frac{1}{m_*+1} = \frac{2}{m_*^2-1} \text{ or } t > \frac{m_*^2-1}{2}$$

2 - case: if $\frac{k_1}{t} = \frac{1}{m_*}$ then (since $k_1 \neq k_3$)

$$\frac{1}{t} \leq \frac{1}{m_* - 2} - \frac{1}{m_*} = \frac{2}{m_*^2 - 2m_*} \quad \text{or } t \geq \frac{m_*^2 - 2m_*}{2}$$

This always means $g'(m) = t \geq \frac{m_*^2 - 2m_*}{2}$ □

Corollary 1. $f'(n) \leq 4 + 2\sqrt{2n+1}$.

Proof. If $f'(n) = m$ then $A'_n(G) = CH(G)$, whenever $diam(G) \leq m$. We have (similarly as in the proof of Lemma 1) $n \geq \frac{m_*^2 - 2m_*}{2}$, that is $m_* \leq 1 + \sqrt{2n+1}$ or $m = f(n) \leq 4 + 2\sqrt{2n+1}$ (where $m_* = [\frac{m-1}{2}]$) □

Lemma 2. *$f(n) \geq m$ iff for each of the intervals $I = [\arctan \frac{c}{d}, \arctan \frac{a}{b}]$ with $a \leq b$, $c \leq d$, $b + d \leq m$ there exists direction $\alpha_i = \arctan \frac{i}{n} \in dir'(n)$ such that $\alpha_i \in I$.*

Proof.

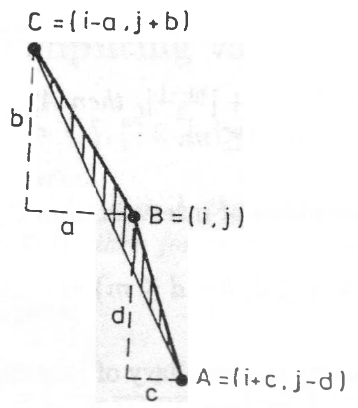

Fig.2

Let $a \leq b, c \leq d, b + d \leq m$ (See Fig. 2). If $f(n) \geq m$ then $B \in Ex'_\alpha(\{A, B, C\})$ But $B \in Ex'_\alpha(G)$ if there exists $\alpha \in dir'(n)$ such that $\alpha \in [\arctan \frac{c}{d}, \arctan \frac{a}{b}]$

The other direction:
Since:

$$\arctan \frac{k}{n} \in [\arctan \frac{c}{d}, \arctan \frac{a}{b}] \Leftrightarrow \arctan \frac{n}{k} \in [\arctan \frac{b}{a}, \arctan \frac{d}{c}]$$

and

$$-\arctan \frac{k}{n} \in [\arctan(-\frac{c}{d}), \arctan(-\frac{a}{b})] \Leftrightarrow \arctan \frac{k}{n} \in [\arctan \frac{a}{b}, \arctan \frac{c}{d}]$$

the statement follows. (Otherwise set G with $diam(G) \leq m$ and $A_n(G) \neq CH(G)$ would be easily conctructed (in contradiction with $f(n) \geq m$).) □

Corollary 2. *Let the family of intervals*

$$J_1 = \{[\frac{c}{d}, \frac{a}{b}],\ a \leq b,\ c \leq d,\ b + d \leq m\} \text{ be given.}$$

Then $f'(n) \leq m$ iff for each interval I_1 from J_1 there exists an integer $k(I_1) \in \{0, \dots, n\}$ such that $k(I_1)/n \in J_1$.

Proof.
Directly from the monotonicity of function $\arctan x$. □

Lemma 3. *If $n \geq [\frac{m-1}{2}]^2 + [\frac{m-1}{2}]$,then $A'_n(G) = CH(G)$, for all the grid point sets G with $diam(G) \leq m$.*

Proof. The minimal width of intervals from J_1 is :

$$\min\{|\frac{a}{b} - \frac{c}{d}|\ |a \leq b,\ c \leq d,\ b + d \leq m\} = \frac{1}{m_*^2 + m_*} \quad (\text{where } m_* = [\frac{m-1}{2}]\ )$$

if $\frac{1}{n} \leq \frac{1}{m_*^2 + m_*}$ (by using the corollary of Lemma 2) follows: $A'_n(G) = CH(G)$ (for the grid point sets G with $diam(G) \leq m$) □

Corollary 3.

$$g'(m) \leq [\frac{m-1}{2}]^2 + [\frac{m-1}{2}]$$

Proof. Follows from the definition of function g'. □

Lemma 4.

$$f'(n) \geq \sqrt{4n+1}\,.$$

Proof. From Lemma 3 for set G with $diam(G) \leq f(n)+1$:
$A'_t(G) = CH(G)$ whenever $t \geq [\frac{f(n)}{2}]^2 + [\frac{f(n)}{2}]$

On the other hand, there must exist a grid point set G, such that $diam(G) = f(n)+1$ and $A'_n(G) \neq CH(G)$ (from the definition of function f'). This means:

$$n < [\frac{f(n)}{2}]^2 + [\frac{f(n)}{2}]\,; \text{ or } f(n) \geq \sqrt{4n+1} \qquad \square$$

Theorem 1.

$$f'(n) = \Theta(\sqrt{n})$$

and

$$g'(m) = \Theta(m^2).$$

Proof. Lemma 1, Corollary of Lemma 1, Corollary of Lemma 3, and Lemma 4. □

3. Algorithm for computing values of function f'

Let the family of intervals $J_2 = \{[\frac{c}{d}, \frac{a}{b}] \in J_1 \mid ad - bc < 3\}$ be given.

Lemma 5. *If for each interval I_2 from J_2 there exists integer $k(I_2) \in \{0,\ldots,n\}$ such that $k(I_2)/n \in I_2$, then for $m \geq 17$ $f(n) \leq m$.*

Proof. Consider intervals from J_2

$$I_2^1 = [\frac{1}{m_*+1}, \frac{1}{m_*}]\,,\; I_2^2 = [\frac{1}{m_*}, \frac{1}{m_*-1}]\,,\; I_2^3 = [\frac{1}{m_*-1}, \frac{1}{m_*-2}]\,,$$

if integers $k(I_2^1), k(I_2^2), k(I_2^3)$, exist, we have $\frac{1}{n} \leq \frac{2}{m_*^2-2m}$ (see the proof of Lemma 1)
Let interval $I = [\frac{c}{d},\ \frac{a}{b}] \in J_1 \setminus J_2$ be given. The width of this interval

is $\frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd} \geq \frac{3}{bd} \geq \frac{3}{m_*^2+m_*}$ However, for $m_* \geq 8(m = 17, 18\ldots)$: $\frac{3}{m_*^2+m_*} \geq \frac{2}{m_*^2-2m_*} \geq \frac{1}{n}$ Thus, for each interval I from $J_1 \setminus J_2$ integer $k(I) \in \{0, \ldots, n\}$ must exist, and using the Corollary of Lemma 2 $f(n) \geq m$ follows for $m \geq 17$. □

Let a family of intervals

$$J_3 = \{[\frac{c}{d}, \frac{a}{b}] \in J_2 \mid ad - bc = 1\} \text{be given.}$$

Lemma 6. *If for each interval I_3 from J_3 there exists an integer $k(I_3) \in \{0, \ldots, n\}$ such thah $k(I_3)/n \in I_3$, then $f(n) \geq m$ (for $m \geq 17$).*

Proof. Let us show that for each interval I from $J_2 \setminus J_3$, there exists an interval $I_3 \in J_3$ such that $I_3 \subset I$. Then, using the previous lemma the statement follows. Let (a, b) denote the largest common divisor for integers a, b. If $(a, b) = 1$, then x, y exist so that $ay - bx = 1$ and $y \leq b$, $x \leq a$. Let $I = [\frac{c}{d}, \frac{a}{b}]$ and $(a, b) = (c, d) = 1$

Since $b + d \leq m$, then either $b \leq m/2$ or $d \leq m/2$

Case 1 $b \leq m/2$ then

$$\left.\begin{array}{l} ad - bc = 2 \\ ay - bx = 1 \end{array}\right\} \Rightarrow a(d-y) - b(c-x) = 1 \ (x, y \text{ exist})$$

if $d \leq y$ then

$$\frac{a}{b} - \frac{c}{d} = \frac{2}{bd} > \frac{1}{by} = \frac{a}{b} - \frac{x}{y}$$

This means that $I_3 = [\frac{x}{y}, \frac{a}{b}] \subset [\frac{c}{d}, \frac{a}{b}] = I$, clearly $I_3 \in J_3$.

If $d > y$ then $c > x$, but the inequalites

$$\frac{a}{b} - \frac{c}{d} < \frac{a}{b} - \frac{x}{y} \text{ and } \frac{a}{b} - \frac{c}{d} < \frac{a}{b} - \frac{c-x}{d-y}$$

can not be simultaneously true. So, one of the intervals

$$I_3' = [\frac{x}{y}, \frac{a}{b}] \text{ or } I_3'' = [\frac{c-x}{d-y}, \frac{a}{b}]$$

from family J_3 is the subinterval of I.

Case 2 $d \leq m/2$

$$\left.\begin{array}{l} xd - cy = 1 \\ ad - bc = 2 \end{array}\right\} \Rightarrow (a-x)d - c(b-y) = 1 (x \leq c,\ y \leq d \text{ exist}).$$

if $a \leq x$ then $b \leq y$, but analogously by the previous,

$$I_3 = [\frac{c}{d}, \frac{x}{y}] \subset I = [\frac{c}{d}, \frac{a}{b}]$$

if $c > x$, then $b > y$, but the one of intervals from J_3
$I_3''' = [\frac{c}{d}, \frac{x}{y}]$ or $I'''' = [\frac{c}{d}, \frac{a-x}{b-y}]$ is subinterval of I. □

Theorem 2. *Let $m \geq 17$. Then: $f(n) \geq m$ iff for each interval I_3 from J_3 there exists an integer $k(I_3) \in [0, \dots, n]$ such that $k(I_3)/n \in I_3$.*

Proof. Lemma 6 and the Corollary of Lemma 2. □

By using Theorem 2, we can give a simple algorithm for computing the values of function f'.
It is clear that: if $(b, m-b) = 1$ then there exists only one number a and only one number c such that: $a(m-b) - cb = 1$ and $c \leq b$, $a \leq m-b$.
Denote: $c = c(b,m)$ and $a = a(b,m)$.
The numbers $c(b,m)$ and $a = a(b,m)$ can be determined in $\mathcal{O}(\lg m)$ time (by using a Eulidean algorithm).
Let: $I(m) = \{[\frac{x}{m-b}, \frac{y}{b}],\ x \leq m-b,\ y \leq b,\ b = 1, \dots m-1\}$ for $m < 17$ and $I(m) = \{[\frac{c(m,b)}{m-b}, \frac{a(b,m)}{b}],\ b = 1, \dots m-1, (b, m-b) = 1\}$ for $m \geq 17$.

ALGORITHM 1
m=0
REPEAT m=m+1

UNTIL there exists an interval from $I(m)$ which does not contain a point of the form $\frac{k}{n}$.
END REPEAT
$f'(n) = m - 1$.
END ALGORITHM 1

For each b we check if $(b, m-b) = 1$ (*in* $\mathcal{O}(\lg m)$ time) and if truly the numbers $a(b,m), c(b,m)$ can be determined in $\mathcal{O}(\lg m)$ time. Therefore, for checking $\mathcal{O}(m \cdot lgm)$ is needed for $m \geq 17$, (for $m > 17$, $\mathcal{O}(1)$ time is needed).
Algorithm 1 takes $\mathcal{O}(m^2 \lg m)$ time to compute $f'(n)$. Using Theorem 1 we derive:

Lemma 7. *By using ALGORITHM 1 $f'(n)$ can be computed in $\mathcal{O}(n \cdot lgn)$ time.*

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REZIME

APROKSIMACIJE KONVEKSNOG OMOTAČA - OCENA USPEŠNOSTI

Ovaj rad sadrži neke ocene uspešnosti jednog aproksimativnog algoritma za konveksne omotače skupova sa konačno mnogo celobrojnih tačaka koje je predložio Stojmenović. Takodje je dat algoritam za odredjivanje vrednosti funkcije $f'(n)$ koja na prirodan način karakteriše uspešnost posmatranog algoritma.

Received by the editors February 21, 1990