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COMMUTATIVITY OF RINGS WITH SOME CONSTRAINTS ON A SUBSET

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Abstract

Let R be a ring and $A(R)$ be an appropriate subset of R. In this paper, it is shown that R is commutative if and only if for every $x, y \in R$, there exist integers $m = m(x,y) > 1$, $n = n(x,y) \geq 0$ such that $[x, x^n y + y^m x] = 0$ and for each $x \in R$ either $x \in Z(R)$, the center of R, or there exists a polynomial $f(t) \in \mathbb{Z}[t]$ such that $x - x^2 f(x) \in A(R)$, where $A(R)$ is a nil commutative subset of R. If R is a left or right s-unital ring, then the following are equivalent: (i) R is commutative. (ii) For every $x, y \in R$, there exist integers $m = m(x,y) > 1$, $n = n(x,y) \geq 0$ such that $[x, x^n y + y^m x] = 0$ and for each $x \in R$ either $x \in Z(R)$ or there exists a polynomial $f(t) \in \mathbb{Z}[t]$ such that $x - x^2 f(x) \in A(R)$, where $A(R)$ is a nil subset of R. (iii) For each $y \in R$, there exists an integer $m = m(y) > 1$ such that $[x, x^n y + y^m x] = 0 = [x, x^n y^m + y^{m^2} x]$ for all $x \in R$, where $n \neq 1$ is a fixed non-negative integer.

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Throughout this paper, R will represent an associative ring (may be without unity 1). Let $Z(R)$ denote the center of R, $N(R)$ the set of nilpotent elements of R, $C(R)$ the commutator ideal of $R, A(R)$ a non-empty subset of R, and $V_R(A(R))$ the centralizer of a subset $A(R)$ of R. Let $\mathbf{Z}[t]$ stand for the totality of polynomials in t with coefficients in $\mathbf{Z}$, the ring of integers. For any x, y in R, we set $[x, y] = xy - yx$.

A ring R is called left (resp. right) s-unital if $x \in Rx$ (resp. $x \in xR$) for each x in R. Further, R is called s-unital if it is both left and right s-unital, that is $x \in xR \cap Rx$ for all x in R. If R is s-unital (resp. left or right s-unital), then for any finite subset F of R there exists an element e in R such that $ex = xe = e$ (resp. $ex = x$ or $xe = x$) for all x in F. Such an element e is called the pseudo (resp. pseudo left or pseudo right) identity of F in R. A ring R is said to be normal if every idempotent element in R is in $Z(R)$.

In this paper, we consider the following ring properties:
$(I - A(R))$ For each $x \in R$, there exists a polynomial $f(t)$ in $\mathbf{Z}[t]$ such that $x - x^2 f(x) \in A(R)$.
$(II - A(R))$ For each $x \in R$, either $x \in Z(R)$, or there exists a polynomial $f(t)$ in $\mathbf{Z}[t]$ such that $x - x^2 f(x) \in A(R)$.
$(III - A(R))$ For every $a \in A(R)$ and $x \in R, [[a, x], x] = 0$.
(IV) For every $x, y \in R$, there exist integers $m = m(x, y) > 1$ and $n = n(x, y) \geq 0$ such that $[x, x^n y + y^m x] = 0$.
(V) For each $y \in R$, there exists an integer $m = m(y) > 1$ such that $[x, x^n y + y^m x] = 0 = [x, x^n y^m + y^{m^2} x]$ for all $x \in R$, where $n \neq 1$ is a fixed non-negative integer.
(VI) For every $x, y \in R$, there exist fixed integers $m > 1$ and $n \geq 0$ such that $[x, x^n y + y^m x] = 0$.

Conjecture 1. *If R is a ring with unity 1 satisfies the property (IV), then R is commutative.*

The well-known Grassman algebra demonstrates that the above conjecture is not true if $m = 1$. Moreover, if we drop the restriction that R has unity 1, in the above conjecture, then the ring R may be badly non-commutative. Indeed, the following example demonstrate this constraint: Let D_k be the ring of all $k \times k$ matrices over a division ring D, and let $A_k = \{ (a_{ij}) \in D_k \mid a_{ij} = 0,\ j \geq i \}$. Then A_k is a non-commutative nilpotent ring of index k, for any positive integer $k > 2$. But A_3 satisfies the polynomial identity $[x, x^n y + y^m x] = 0$ for any integers $m > 1,\ n \geq 0$.

The objective of the present paper is to study the equivalence of the above listed properties with reference to the commutativity of the ring under consideration. Here, we consider some constraints on $A(R)$ which together with (IV) imply commutativity even for those rings which may not have unity. Indeed, we prove the following results.

Theorem 1. *A ring R is commutative if and only if R satisfies (IV) and $(II - A(R))$ for a commutative subset $A(R)$ of $N(R)$.*

Theorem 2. *If R is a left or right s-unital ring, then the following statements are equivalent:*

(i) R is commutative.

(ii) R satisfies (IV) and there exists a subset $A(R)$ of $N(R)$ for which R satisfies $(II - A(R))$.

(iii) R satisfies (V).

In preparation for the proof of our results, we first collect a number of well- known results.

Lemma 1. *([7, Lemma 3]). Let R be a ring such that $[x,[x,y]] = 0$ for all x and y in R. Then $[x^k, y] = kx^{k-1}[x,y]$ for any positive integer k.*

Lemma 2. *([2, Lemma 2]). Let R be a ring with unity 1, and let x and y be elements of R. If $kx^m[x,y] = 0$ and $k(x+1)^m[x,y] = 0$ for some integers $m \geq 1$, and $k \geq 1$, then necessarily $k[x,y] = 0$.*

Lemma 3. *([9]).*

(i) Let Φ be a ring homomorphism of R onto $R^\star$. If R satisfies $(I - A(R)), (II - A(R))$ or $(III - A(R))$ then $R^\star$ satisfies $(I - \Phi(A(R)))$, $(II - \Phi(A(R)))$ or $(III - \Phi(A(R)))$ respectively.

(ii) If $A(R)$ is commutative and R satisfies $(II - A(R))$, then $N(R)$ is commutative nil ideal of R containing $C(R)$ and is contained in $V_R(A(R))$. In particular, $(N(R))^2 \subseteq Z(R)$.

(iii) If there exists a commutative subset $A(R)$ of $N(R)$ for which R satisfies $(II - A(R))$ and $(III - A(R))$, then R is commutative.

Lemma 4. *([10, Lemma]). Let R be a left (resp. right) s-unital ring. If for each pair of elements x and y in R, there exists a positive integer $k = k(x, y)$ and an element $e = e(x, y)$ of R such that $x^k e = x^k y^k e$ and $y^k e = y^k$ (resp. $ex^k = x^k$ and $ey^k = y^k$), then R is s-unital.*

Lemma 5. *([8, Lemma 3]). Let R be a ring with unity, and let k and m be natural numbers. If $(1 - y^k)x = 0$, then $(1 - y^{km})x = 0$ for all $x, y \in R$.*

Theorem 3. *([6, Theorem]). Let f be a polynomial in n non-commuting indeterminates $x_1, x_2, \ldots, x_n$ with relatively prime integral coefficients. Then the following are equivalent:*

(a) Every ring satisfying the polynomial identity $f = 0$ has a nil commutator ideal.

(b) Every semi-prime ring satisfying $f = 0$ is commutative.

(c) For every prime p, $(GF(p))_2$, the ring of 2×2 matrices over the Galois field $GF(p)$, fails to satisfy $f = 0$.

Theorem 4. *([4, Theorem 3]). If R is a ring with center $Z(R)$ such that for every $a \in R$ there exists a polynomial $p_a(t)$ such that $a - a^2 p_a(t) \in Z(R)$, then R is commutative.*

Theorem 5. *([3, Theorem 19]). Let R be a ring and let $n = n(x) > 1$ be an integer depending on x. If $x^n - x \in Z(R)$ for all $x \in R$, then R is commutative.*

In preparation for the proof of our results, we prove the following:

Lemma 6. *Let R be a ring satisfying (IV). Then R is normal.*

Proof. Given an idempotent element e and an element x in R, then there exist integers $m = m(e, e + ex(1-e)) > 1$ and $n = n(e, e + ex(1-e)) \geq 0$ such that $[e, e^n(e + ex(1-e)) + (e + ex(1-e))^m e] = 0$. This implies that $ex(1-e) = [e, ex(1-e)] = 0$, that is $ex = exe$. Similarly, $xe = exe$. Therefore, $ex = xe$ for all $x \in R$. Thus R is normal.

□

Lemma 7. *Let R be a ring with unity 1 satisfying (IV). Then $N(R) \subseteq Z(R)$.*

Proof. If $a \in N(R)$ and $x \in R$, then there exist integers $m_1 = m(x,a) > 1$ and $n_1 = n(x,a) \geq 0$ such that $x^{n_1}[x,a] = [a^{m_1},x]x$ for all $x \in R$. If $m_2 = m(x,a^{m_1}) > 1$ and $n_2 = n(x,a^{m_1}) \geq 0$, then $x^{n_2}[x,a^{m_1}] = [(a^{m_1})^{m_2},x]x = [a^{m_1 m_2},x]x$ for all$x \in R$. Thus $x^{n_1+n_2}[x,a] = -[a^{m_1 m_2},x]x^2$ for all $x \in R$. Hence, for any positive integer t,

$$x^{n_1+n_2+\ldots+n_t}[x,a] = (-1)^{t-1}[a^{m_1 m_2 \ldots m_t},x]x^t \text{for all } x \in R.$$

But a is nilpotent, then $a^{m_1 m_2 \ldots m_t} = 0$ for sufficiently large t. So

$$x^{n_1+n_2+\ldots+n_t}[x,a] = 0 \text{ for all } x \in R.$$

Let $n'(x) = n_1 + n_2 + \ldots + n_t$. Then $x^{n'(x)}[x,a] = 0$. For $n' = \max\{n'(x), n'(x+1)\}$, we have $x^{n'}[x,a] = 0$ and $(x+1)^{n'}[x,a] = 0$, which by Lemma 2 yields $[x,a] = 0$ for all $x \in R$. Therefore, $a \in Z(R)$ and thus $N(R) \subseteq Z(R)$.

□

By using Theorem 4, we conclude the following:

Theorem 6. *Let R be a ring with unity 1 satisfying (IV) and $(II - A(R))$ for a subset $A(R)$ of $N(R)$. Then R is commutative.*

Lemma 8. *Let R be a ring with unity 1 satisfying (V). Then $C(R) \subseteq Z(R)$.*

Proof. Let $n \geq 0$ be a fixed integer. Then for any $y \in R$, there exists an integer $m = m(y) > 1$ such that condition (V) can be rewritten as

(1) $$x^n[x,y] = [y^m,x]x \text{ for all } x \in R,$$

and

(2) $$x^n[x,y^m] = [y^{m^2},x]x \text{ for all } x \in R.$$

This implies that $(x+1)^n[x,y]x = [y^m,x](x+1)x = x^n[x,y](x+1)$ for all $x,y \in R$. By Theorem 3, we observe that $C(R)$ is a nil ideal, that is

$C(R) \subseteq N(R)$, since $x = e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and $y = e_{21} + e_{22} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ fail to satisfy the identity $(x+1)^n[x,y]x - x^n[x,y](x+1) = 0$ in $(GF(p))_2$, for a prime p. Hence, by Lemma 6, $C(R) \subseteq N(R) \subseteq Z(R)$.

□

Lemma 9. *Let R be a ring with unity 1 satisfying (V). Then R is commutative.*

Proof. For $n = 0$, we get $[x,y] = [y^m,x]x$ for all $x,y \in R$. Replace x by $x+1$ to obtain $[y^m,x] = 0$ for all $x,y \in R$. Thus $[x,y] = [y^m,x]x = 0$ for all $x,y \in R$. Therefore, R is commutative.

Now, we suppose that $n > 1$. Let $t = 2^{n+1} - 2^2$. Then $t > 0$ for $n > 1$. By using (1), we get

$$\begin{aligned} tx^n[x,y] &= (2^{n+1} - 2^2)x^n[x,y]] \\ &= (2x)^n[(2x),y] - 2^2x^n[x,y] \\ &= (2x)^n[(2x),y] - [y^m,(2x)](2x) \\ &= 0. \end{aligned}$$

The usual argument of replacing x by $x+1$, Lemma 2 gives $t[x,y] = 0$. Again, Lemma 1 and Lemma 8 together imply that $[x^t,y] = tx^{t-1}[x,y] = 0$ for all x and y in R. So $x^t \in Z(R)$ for all $x \in R$.

Further, using (1), (2) and the fact that $C(R) \subseteq Z(R)$ (see Lemma 7), we see that

$$\begin{aligned} (1 - y^{(m-1)^2})[x,y]x^{2n-1} &= [x,y]x^{2n-1} - y^{(m-1)^2}[x,y]x^{2n-1} \\ &= [y^m,x]x^n - y^{(m-1)^2}[y^m,x]x^n \\ &= -x^n[x,y^m] - my^{m-1}y^{(m-1)^2}[y,x]x^n \\ &= -x^n[x,y^m] + my^{m(m-1)}x^n[x,y] \\ &= -x^n[x,y^m] + my^{m(m-1)}[y^m,x]x \\ &= -x^n[x,y^m] + [y^{m^2},x]x \\ &= 0. \end{aligned}$$

Replacing x by $x+1$ gives $(1 - y^{(m-1)^2})[x,y](x+1)^{2n-1} = 0$ for all $x,y \in R$. By Lemma 2, we obtain

$$(3) \qquad (1 - y^{(m-1)^2})[x,y] = 0 \text{ for all } x,y \in R.$$

But since, $x^t \in Z(R)$, for all $x \in R$, we get

$$[x, y - y^{t(m-1)^2+1}] = (1 - y^{t(m-1)^2})[x, y] = 0 \text{ for all } x, y \in R.$$

Thus $y - y^{t(m-1)^2+1} \in Z(R)$, for $m = m(y) > 1$. Therefore, R is commutative Theorem 5.

□

Now, we are in a position to prove our results.

Proof of Theorem 1. It is straightforward to see that a commutative ring R satisfies the conditions given in the theorem.

Now, let R be a ring satisfying the hypothesis of our theorem. If R has unity 1, then the result follows from the Theorem 6. So we suppose that R does not contain unity 1. In view of Lemma 3 (i), R can be assumed to be a subdirectly irreducible ring without unity 1. Let $x \in R \backslash Z(R)$ be an arbitrary element. By hypothesis, R satisfies $(II - A(R))$ for a commutative subset $A(R)$ of $N(R)$, and thus, there exists an element $y \in < x >$, the subring generated by x, and an integer $m > 1$ such that $x^m = x^{m+1}y$. Clearly, $e = x^m y^m$ is idempotent with $x^m = x^m e$, and also e is central by Lemma 5. Since R has no unity , $e = 0$. Again by Lemma 3 (ii), x is in the commutative ideal $N(R)$ and $[x, [x, a]] = 0$ for all $a \in A(R)$. Hence R is commutative by Lemma 3 (iii).

□

Proof of Theorem 2. If R is a commutative left or right s-unital ring, then clearly, R satisfies (ii) and (iii).

Now, suppose that R satisfies (ii). First, we show that R is s-unital. Let R be a right s-unital ring, and let x and y be arbitrary elements of R. Then we can find an element $e \in R$ such that $xe = x$ and $ye = y$. Further, there exist integers $m = m(x, e) > 1$ and $n = n(x, e) \geq 0$ such that

$$e^m x^2 = -[x, x^n e + e^m x] + x^2 = x^2.$$

Similarly, there exist integers $m' = m'(y, e) > 1$ and $n' = n'(y, e) \geq 0$ such that $e^{m'} y^2 = y^2$. Therefore, $e^{mm'} x^2 = x^2$ and $e^{mm'} y^2 = y^2$. Hence, R is s-unital by Lemma 4.

Now, suppose that R is a left s-unital ring. Let $x, y \in R$. Then there exist an element $e \in R$ such that $ex = x$ and $ey = y$. Also, there exists integers $m = m(x,e) > 1$ and $n = n(x,e) \geq 0$ such that $x^{n+1}e = [x, x^n e + e^m x] + x^{n+1} = x^{n+1}$. Similarly, if $m' = m(y,e) > 1$ and $n' = n(y,e) \geq 0$, then we have $y^{n'+1}e = y^{n'+1}$. Hence, $x^{n+n'+1}e = x^{n+n'+1}$, and $y^{n+n'+1}e = y^{n+n'+1}$. Again, by Lemma 4, R is an s-unital ring.

In view of Proposition 1 of [5], we may assume that R has unity 1. Hence R is commutative by Theorem 6. Thus (ii) implies (i).

Finally, If R satisfies (iii), then as argued above, we may assume that R has unity 1. Hence, R is commutative by Lemma 9.

□

As a consequence of Theorem 2, we have

Theorem 7. *Let R be a left or right s-unital ring satisfying (VI). Then R is commutative.*

Proof. If $n = 1$, then $x[x,y] = [y^m, x]x$ for all $x, y \in R$. Replace x by $x+1$ to get $[x,y] = [y^m, x]$ for all $x, y \in R$. Thus R is commutative by [1, Theorem 1]. Let $n \neq 1$ be non-negative integer. Then R is commutative by Theorem 2.

□

Example 1. Let R be an algebra over $GF(2)$ of dimension 4 with $\{1, a, b, c\}$ as a basis which also satisfies the multiplication rule

$$a^2 = 1 + a, ab = c, ca = b, ac = ba = b + c, \text{ and } bc = cb = b^2 = c^2 = 0.$$

Then R becomes a non-commutative ring whose nilpotent elements commute among themselves. Let $A(R) = N(R)$ which is a commutative subset of R. Then for any $x \in R$, we see that $x - x^4 = x - x^2(x^2) \in N(R)$. Thus R satisfies $(II - A(R))$. So R fails to be commutative if it does not satisfy (IV). Thus the condition (IV) is essential for commutativity of the ring in the hypothesis of Theorem 1, even if R has unity 1.

Example 2. Theorem 1 need not be true if we drop the condition that $A(R)$ is commutative. For this, consider

$$R = \{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} a, b, c \in GF(2)\}.$$

Then R is a nilpotent ring of index 3 and also $N(R) = R$. Further , R satisfies (IV). However, with $A(R) = N(R)$, R also satisfies $(II - A(R))$. But R is not commutative.

Remark 1. Example 2 also shows that Theorem 2 can not be extended for arbitrary rings.

Example 3. This example shows that both conditions (IV) and $(II-A(R))$ in Theorem 2 (ii) are essential for the ring R with unity 1 to be commutative. Let

$$R = \{aI + S : S = \begin{pmatrix} 0 & b & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix} \text{ and}$$

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : a, b, c, d \in GF(2)\}.$$

Then, it is easy to check that $N(R) = S$, and R does not satisfy (IV). Let $A(R) = N(R)$. Then for all $x \in R$, we have $x - x^2 f(x) \in A(R)$. However, R is not commutative.

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REZIME

KOMUTATIVNOST PRSTENA SA NEKIM OGRANIČENJIMA NAD ODREDJENIM PODSKUPOM

Neka je R prsten i $A(R)$ odredjeni podskup od R. U ovom radu pokazano je da je R komutativan ako i samo ako za svako $x, y \in R$, postoje celi brojevi $m = m(x,y) > 1$, $n = n(x,y) \geq 0$ tako da je $[x, x^n y + y^m x] = 0$ i za svako $x \in R$ ili je $x \in Z(R)$, centar od R, ili postoji polinom $f(t) \in \mathbf{Z}[t]$ takav da je $x - x^2 f(x) \in A(R)$, gde je $A(R)$ jedan nil komutativan podskup od R. Ako je R levo ili desno s -unitalan prsten, tada važe ekvivalencije: (i) R je komutativan. (ii) Za svako $x, y \in R$, postoje celi brojevi $m = m(x,y) > 1$, $n = n(x,y) \geq 0$ tako da je $[x, x^n y + y^m x] = 0$ i za svako $x \in R$ ili je $x \in Z(R)$ ili postoji polinom $f(t) \in \mathbf{Z}[t]$ tako da je $x - x^2 f(x) \in A(R)$, gde je $A(R)$ nil podskup od R. (iii) Za svako $y \in R$, postoji ceo broj $m = m(y) > 1$ tako da

je $[x, x^n y + y^m x] = 0 = [x, x^n y^m + y^{m^2} x]$ za sve $x \in R$, gde je $n \neq 1$ fiksiran nenegativan ceo broj.

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