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# ON THE NUMBER OF LINEAR PARTITIONS

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**Abstract**

In this paper a formula is derived for the number of linear partitions of a given point set $S$ in three-dimensional space, depending on the configuration formed by the points of $S$.

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## 1. Introduction

A $k$–set is a set of cardinality $k$. The cardinality of a set $A$ is denoted by $\mid A \mid$. Given a finite point set $S$ in an $n$–dimensional space, a linear partition of $S$ is a partition of $S$ into subsets $X$ and $S\setminus X$, which is induced by an $(n-1)$–dimensional hyperplane $H$: it is assumed that the intersection of $S$ and $H$ is empty and that the sets $X$ and $S\setminus X$ respectively belong to distinct half-spaces $w.r.t.H$. The enumeration problem for linear partitions is closely related to the efficiency measurement problem for linear discriminant functions in pattern recognition ([1]) and to the many others algorithmic problems ([2]).

## 2. Two-dimensional case

An unordered pair $(A, B)$ of distinct points belonging to a finite point-set $S$ in a plane is said to be **minimal** ($w.r.t.S$) if there does not exist a third point $C$ of $S$, which belongs to the open line-segment $AB$.

The following statement might be very helpful for the enumeration of linear partitions of a planar set $S$. The number of linear partitions of a planar point set $S$ and the number of minimal pairs of $S$ are denoted by $LP(S,2)$ and $mp(S)$ respectively.

The following statement is well known.

**Lemma 1.**

$$LP(S,2) = mp(s) + 1. \tag{1}$$

*If $S$ is a well-distributed point set (no three points colinear), then every pair of points in $S$ is a minimal pair and (2) becomes:*

$$LP(S,2) = 1 + \binom{N-1}{1} + \binom{N-1}{2} \tag{2}$$

From Lemma 1 the following statement can be easily deduced.

**Theorem 1.** *Let $p_1, p_2, ..., p_k$ be all lines determined by a planar point set $S$ (each line contains at least two points of $S$ and let $s_i(1)$ denote the number of points of $S$ belonging to the line $p_i$ $(i = 1, 2, ..., k)$. Then*

$$LP(S,2) = 1 + \binom{N-1}{1} + \binom{N-1}{2} - \sum_{i=1}^{k} \binom{s_i(1)-1}{2}. \tag{3}$$

*Proof.* If the line $p_i$ contains $s_i(1)$ points from $S$, then they determine $\binom{s_i(1)}{2}$ pairs among which exactly $s_i(1) - 1$ are minimal pairs; so the number of pairs which are not minimal is

$$\binom{s_i(1)}{2} - (s_i(1) - 1) = \binom{s_i(1)-1}{2}.$$

Then the number of non-minimal pairs of points in $S$ is $\sum_{i=1}^{k} \binom{s_i(1)-1}{2}$.

Now, the statement follows from the fact that the total number of pairs in $S$ is $\binom{N-1}{1} + \binom{N-1}{2}$. □

## 3. A generalization of the two-dimensional case (multiple points permited)

Before proceedeng with the two-dimensional case, let us consider the very special case of one dimension.

**Theorem 2.** *If $S$ is a set of points in $\mathbf{R}_1$, $|S| = N$, then*

$$LP(S,1) = 1 + \binom{N-1}{1} = N.$$

*Proof.* Obvious . □

Now, we consider a generalization in $\mathbf{R}_1$, when some points coincides (multiplicity of points ).

By $A_i^j$ we denote a point with multiplicity $j$, i.e., $j$ points which coincide.

**Theorem 3.** *Let $S$ consists of*

$$A_1^{s_1(0)}, A_2^{s_2(0)}, \dots, A_k^{s_k(0)},$$

*where*

$$s_1(0) + s_2(0) + \dots + s_k(0) = N.$$

*Then*

$$\overline{LP}(S,1) = 1 + \binom{N-1}{1} - \sum_{i=1}^{k} \binom{s_i(0)-1}{1}, \tag{4}$$

*where $\overline{LP}(S,1)$ is the number of linear partitions of the set $S$.*

*Proof.* Obvious. □

Let $S \subset R^2$, $| S |= N$, consists of points

$$A_1^{s_1(0)}, A_2^{s_2(0)}, \ldots, A_{k_0}^{S_{k_0^s}(0)},$$

where $s_i(0)$ is the multiplicity of the point $A_i$ $(i = 1, 2, \ldots, k_0)$, and

$$s_1(0) + s_2(0) + \ldots + s_{k_0}^s(0) = N.$$

Let

(5) $$p_1, p_2, \ldots, p_{k_1^s}$$

be all different lines determined by the points of $S$ (eash line contains at least two non-coicident points of $S$ ).

Denote by $s_i^s(1)$ the number of points of $S$ belonging to the line $p_i$ (with the corresponding multiplicities): $i = 1, 2, \ldots, k_1^s$. Let $r_i^s(0)$ be the number of different lines from (5) through the point $A_1^{s_i(0)}$ $(i = 1, 2, \ldots, k_0^s)$.

If we denote by $\overline{LP}(S, 2)$ the number of linear partitions of the set $S$, then the following statement can be proved.

**Theorem 4.**

$$\overline{LP}(S,2) = 1 + \binom{N-1}{1} + \binom{N-1}{2} - \sum_{i=1}^{k_0^s} \binom{s_i(0)-1}{1} -$$

(6) $$-\sum_{i=1}^{k_1^s} \binom{s_i^s(1)-1}{2} + \sum_{i=1}^{k_0} (r_i^s(0) - 1) \binom{s_i^s(0)-1}{2}.$$

*Proof.* The proof will be by induction on $N$. Consider a set of points $S, | S |= N + 1$. Let $A$ be a single point (with multiplicity one), of $S$, which is a vertex of the convex hull of $S$. By induction hypothesis, for the set $X = S \setminus \{A\}$ we have

(7) $$LP(X, 2) = LP(S \setminus \{A\}) = 1 + \binom{N-1}{1} + \binom{N-1}{2} -$$

$$-\sum_{i=1}^{k_0^X} \binom{s_i^X(0)-1}{1} - \sum_{i=1}^{k_1^X} \binom{s_i^X(1)-1}{2} + \sum_{i=1}^{k_0^X} (r_i^X - 1) \binom{s_i^X(0)-1}{2}.$$

Let us determine the number of movable line partitions of $X$.

Project $X$ from the point $A$ into a line $a$ which separate $A$ from $X$. Let $X'$ be the projection of $X$. The number of movable line partitions of $X$ is, according to Theorem 3.

$$\overline{LP}(X',1) = 1 + \binom{N-1}{1} - \sum \binom{s_i^{X'}(0)-1}{1}. \tag{8}$$

From (7) and (8) it follows that

$$LP(S,2) = LP(X,2) + LP(X',1) = 1 + \binom{N}{1} + \binom{N}{2} -$$

$$-\sum_{i=1}^{k_0^s} \binom{s_i^s(0)-1}{1} - \sum_{i=1}^{k_1^s} \binom{s_i^s(1)-1}{2} + \sum_{i=1}^{k_0^s}(r_i^s(0)-1)\binom{s_i^s(0)-1}{2}.$$

In the case when none of the vertices of the convex hull of $S$ is a single point, we shall see what happen when the multiplicity of one point $A \in X$, $|X| = N$, increases by one, producing the set $S$.

Denote, for the sake of simplicity, the sums from the righthand side of (7) by $\sum_0^X$, $\sum_1^X$, $\sum_2^X$ respectively and those from (8) by $\sum_0^s$, $\sum_1^s$ and $\sum_2^s$.

Since

$$1 + \binom{N}{1} + \binom{N}{2} - (1 + \binom{N-1}{1} - \binom{N-1}{2}) = N,$$

we need to prove that

$$\sum_0^s + \sum_1^s - \sum_2^s) - (\sum_0^X + \sum_1^X - \sum_2^X) = N. \tag{9}$$

Let we have the point $A_i^{s_i(0)+1}$ in $S$ instead of $A_i^{s_i(0)}$ in $X$. Then

$$\sum_0^s - \sum_0^X = 1. \tag{10}$$

Taking into account that $r_i^s(0) = r_i^x(0)$ and $s_i^s(0) = s_i^X(0) + 1$, we have

$$\sum_2^s - \sum_2^X = (r_i^X(0)-1)\binom{s_i^X(0)}{2} - \binom{s_i^X(0)-1}{2},$$

i.e.,

(11) $$\sum_2^s - \sum_2^X = (r_i^X(0)-1)(s_i^X(0)-1).$$

Let $\{p_{j_1}, p_{j_2}, ..., p_{j_{r_i}}\}$ be the set of lines from (5) through the point $A$. Then

$$\sum_2^s - \sum_2^X = \sum_{i=1}^{r_i^X(0)} (\binom{s_{j_i}^s(1)}{2} - \binom{s_{j_i}^s(1)-1}{2}) =$$

$$\sum_{i=1}^{r_i^X(0)} (s_{j_i}^s(1)-1) = \sum_{i=1}^{r_i} s_{j_i}^s(1) - r_i^X(0) =$$

$$= N - s_i^X(0) + r_i^X(0)s_i^X(0) - r_i^X(0) =$$

$$= N + r_i^X(0)(s_i^X(0)-1) - s_i^X(0),$$

and finally,

(12) $$\sum_2^s - \sum_2^X = N + (r_i^X(0)-1)(s_i(0)-1) - 1.$$

Now, (9) follows from (10), (11) and (12) . □

## 4. Three-dimensional case

Let $S$ be the set of different points in three-dimensional space, $|S| = N$. Suppose that

(13) $$p_1, p_2, ..., p_{k_1}^s$$

are all different lines determined by points of $S$ (each line contains at least two different points of $(S)$, and

(14) $$R_1, R_2, ..., R_{k_2}^s$$

are all different planes determined by points of $S$ (each plain contains at least three different points of $S$).

We introduce the following notation:

$s_i^s(1)$ - the number of points of $S$ lying on $p_i$;

$s_i^s(2)$ - the number of points of $S$ lying in $R_i$;

$r_i^s(1)$ - the number of planes from (7) containing the line $p_i$.

If we denote by $LP(S,3)$ the number of linear partitions of the set $S$ then the following statement can be proved.

**Theorem 5.**

$$LP(S,3) = 1 + \binom{N-1}{1} + \binom{N-1}{2} + \binom{N-1}{3} - \sum_{i=1}^{k_1^s} \binom{s_i^s(1)-1}{2} -$$

$$(15) \qquad - \sum_{i=1}^{k_2^s} \binom{s_i^s(2)-1}{3} + \sum_{i=1}^{k_1^s} (r_i^s(1)-1) \binom{s_i^s(1)-1}{3}.$$

*Proof.* We use induction (with trivial basis) on $|S|$. Suppose that the statement is valid for $|S| = N$. Consider $|S| = N+1$.

Take a point $A$ of $S$. For the sake of simplicity, we may assume (without any loss of generality) that $A$ is a vertex of the convex hull of $S$. Denote $S \setminus \{A\}$ by $X$. Consider the projection $\pi$ of $X$ onto a plane $\alpha$ separating $A$ from $X$, point $A$ being the center of projection. Those linear partitions of $X$ which can be established by using planes through the point $A$ are said be **movable** (w.r.t. $A$).

The number of additional linear partition which are obtained after extension of the set $X$ to $S$ (by adding the point $A$) is equal to the number of movable (w.r.t. $A$) linear partitions of $X$. This last number is equal to $\overline{LP}(Y,2)$, i.e. to the number of linear partitions of the planar point set $Y = \pi(X)$. The corresponding bijection is established by the projection $\pi$. Namely, each movable linear partitions of $X$ (w.r.t. $A$) may be represented by a plane $H$ through $A$. The line $h = \pi(H)$ corresponds to a linear partition of $Y = \pi(X)$ in $\alpha$. Conversely, given a linear partition of $Y$ with the corresponding line $h$, the plane through $h$ and $A$ determines the associated movable linear partition of $S$.

It follows that

$$(16) \qquad LP(S,3) = LP(X,3) + \overline{LP}(Y,2).$$

Now (15) can be deduced from (16) using induction hypothesis and Theorem 4.

Namely, by induction hypothesis,

$$(17)\qquad LP(X,3) = 1 + \binom{N-1}{1} + \binom{N-1}{2} + \binom{N-1}{3} -$$

$$-\sum_{i=1}^{k_1^X} \binom{s_i^X(1)-1}{2} - \sum_{i=1}^{k_2^X} \binom{s_i^X(2)-1}{3} + \sum_{i=1}^{k_1^X} (r_i^X(1)-1) \binom{s_i^X(1)-1}{3},$$

while, according to Theorem 4,

$$(18)\quad \overline{LP}(Y,2) = 1 + \binom{N-1}{1} + \binom{N-1}{2} - \sum_{i=1}^{k_0^Y} \binom{s_i^Y(0)-1}{1} -$$

$$-\sum_{i=1}^{k_1^Y} \binom{s_i^Y(1)-1}{2} + \sum_{i=1}^{k_0^Y} (r_i^Y(0)-1) \binom{s_i^Y(0)-1}{2}.$$

It can be verified that

$$(19)\qquad \sum_{i=1}^{k_1^X} \binom{s_i^X(1)-1}{2} + \sum_{i=1}^{k_0^Y} \binom{s_i^Y(0)-1}{1} = \sum_{i=1}^{k_1^s} \binom{s_i^s(1)-1}{2},$$

$$(20)\qquad \sum_{i=1}^{k_1^X} \binom{s_i^X(2)-1}{3} + \sum_{i=1}^{k_1^Y} \binom{s_i^Y(1)-1}{2} = \sum_{i=1}^{k_2^s} \binom{s_i^s(2)-1}{3}$$

and

$$(21)\quad \sum_{i=1}^{k_1^X} (r_i^X(1)-1) \binom{s_i^X(1)-1}{3} + \sum_{i=1}^{k_0^Y} (r_i^Y(0)-1) \binom{s_i^Y(0)-1}{3} =$$

$$= \sum_{i=1}^{k_1^s} (r_i^s(1)-1) \binom{s_i^s(1)-1}{3}.$$

Now, the statement follows from (16) - (21) . □

## References

[1] Duda,R.O., Hart, P.E.: Pattern Classification and Scene Analysis, Wiley, New York, 1973.

[2] Edelsbrunner, H.: Algorithms in Combinatorial Geometry, Springer Verlag, Heidelberg, 1987.

**REZIME**

O BROJU LINEARNIH PARTICIJA

U radu je izvedena formula za broj linearnih particija za dati skup tačaka $S$ u trodimenzionalnom prostoru. Taj broj zavisi od kardinalnosti skupa $S$ i od rasporeda tačaka toga skupa.

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