Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 22, 2 (1992), 101-107

Review of Research
Faculty of Science
Mathematics Series

GENERATING FUNCTIONS OF $S_N(X)$ FROM THE VIEW POINT OF LIE-ALGEBRA

M.A. Pathan, Ishart Jahan

Department of Mathematics, Aligarh Muslim University
Aligarh 202 001, India

Abstract

The generating functions for the polynomials $S_N(X)$ introduced recently by W. Schultz-Piszachich [8] in his studies of a certain family of isotropic turbulence of fields, are obtained by using the representation of a Lie-group T_3 [5]. A few special cases of interest are also discussed.

AMS Mathematics Subject Classification (1991): 33C45
Key words and phrases: special functions, Lie algebra.

1. Introduction

Let J_3 be the Lie-algebra of a three dimensional complex local Lie-group T_3, a multiplicative matrix group with elements (cf. Miller [5, p. 10])

$$g = \begin{pmatrix} 1 & 0 & 0 & \tau \\ 0 & e^{-\tau} & 0 & c \\ 0 & 0 & e^{\tau} & b \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad b, c, \tau \in \mathbf{C}. \tag{1.1}$$

T_3 has the topology of $\mathbb{C}^3$ and is simply connected (Pontriagin [6, Chapter 8]). A basis of J_3 is provided by the matrices [5, p. 11]

$$j^+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad j^- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$j^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with commutation relations

$$[j^3, j^+] = j^+, \; [j^3, j^-] = -j^-, \; [j^+, j^-] = \mathcal{O}.$$

The machinery constructed in [5, Chapters 1 and 2] will here be applied to find a realization of the representation $Q(\omega, m_0)$ of J_3 where ω, m_0 are complex constants such that $\omega \neq 0$ and $0 \leq \text{Re}\, m_0 \leq 1$, The spectrum S of this representation is the set $\{m_0 + k : k \text{ an integer}\}$. In particular, we are looking for the functions $f_m(x, y) = Z_m(x)\, e^{my}$, such that

$$J^3 f_m = m f_m, \; J^+ f_m = \omega f_{m+1}, \; J^- f_m = \omega f_{m-1},$$

$$C_{0,0} f_m = J^+ J^- f_m = \omega^2 f_m, \; \omega \neq 0 \tag{1.2}$$

for all $m \in S$, where the differential operators $J^{\pm}$, J^3 are given by

$$\begin{aligned} J^+ &= -xe^y\, \partial/\partial x + 2e^y \partial/\partial y + (x+1)\, e^y, \\ J^3 &= \partial/\partial y, \\ J^- &= -\frac{e^{-y}}{x}\, \partial/\partial x + e^{-y}/x. \end{aligned} \tag{1.3}$$

The operators J^+, J^- and J^3 satisfy the commutation relations

$$[J^3, J^{\pm}] = \pm\, J^{\pm}, \; [J^{\pm}, J^-] = 0.$$

In terms of the functions $Z_m(x) = S_m(x)$, these relations become

$$[-x\frac{d}{dx} + (x + 2m + 1)]\, S_m(x) = S_{m+1}(x)$$

$$(1.4)\qquad [-\frac{1}{x}\frac{d}{dx}+\frac{1}{x}]\,S_m(x)=S_{m-1}(x)$$

$$[-\frac{x}{2m}\frac{d^2}{dx^2}+\frac{(x+m)}{x}\frac{d}{dx}]\,S_m(x)=S_m(x)$$

where the polynomials $S_m(x)$, introduced recently by W. Schultz Piszachich [8] in his investigation of a certain family of isotropic turbulence fields, are defined by

$$(1.5)\qquad S_n(x)=\frac{(2n)!}{2^n\,n!}\sum_{k=0}^{n}\frac{(-n)_k\,(2x)^k}{k!}.$$

These polynomials occur in the following representation of the energy spectral functions (cf. [8, p. 312])

$$(1.6)\qquad E_n(k,t)=\frac{\bar{u}^2 a_n^5}{2^n(n-1)!}k^4\,\exp(-a_n k)\,S_{n-3}\,(a_n k),\ n\geq 3.$$

Srivastava [10] has shown that $S_n(x)$ can be expressed in terms of Laguerre polynomials $L_n^\alpha(x)$ [7], Srivastava-Singhal polynomials $G_n^\alpha(x,r,p,s)$ [9], Bessel polynomials $y_n(x)$ [3] and modified Bessel polynomials $K_n(x)$ [11, p. 168 (32)]. These relations are

$$\begin{aligned} S_n(x) &= n!\,(-2)^{-n}\,L_n^{(-2n-1)}\,(2x)\\ &= (-1)^n\,n!\,G_n^{(-1)}\,(x,1,1,-2)\\ &= x^n\,y_n\,(1/x)\\ &= \sqrt{2/\pi}\,x^{n+1/2}\,e^x\,K_{n+1/2}\,(x). \end{aligned}$$

Many useful and important properties of the polynomials $S_n(x)$ were obtained by U. Werner and W. Pietsh [11] and Srivastava [10].

If functions $Z_m(x)$ defined for each $m\in S$ satisfy (1.4) for $\omega=1$, then the vectors $f_m(x,y)=e^{my}\,Z_m(x)$ form a basis for a realization of the representation $Q(1,m_0)$ of J_3. The differential operators (1.3) generate a Lie-algebra which is the algebra of generalized Lie derivatives of a multiplier representation T of T_3. If C1 is the space of all the functions analytic in some neighborhood of the point $(x^0,y^0)=(1,0)$, the Lie-derivatives (1.3) define a local multiplier representation T of T_3 on C1.

Here, we obtain generating functions of $S_n(x)$ by the Lie-group theoretical approach. The principal interest in our results lies in the fact that a number of special cases would yield inevitably to many new and known results of the theory of special functions.

2. Multiplier representation T

A simple computation using [5, p. 18] and (1.3) gives

$$
\begin{aligned}
&[T(\exp \tau j^3)f](x^0,t^0) = (x^0, e^\tau t^0),\\
&[T(\exp cj^-)f](x^0,t^0) = \exp(x-(x^{02}-2c/t^0)^{1/2})\cdot f(\sqrt{x^{02}-2c/t^0}, t^0),\\
&[T(\exp bj^+)f](x^0,t^0) = \exp(-x^0 t^{01/2}(-2b+1/t^0)^{1/2}+x^0-\log t^{01/2})\cdot\\
&\cdot(-2b+1/t^0)^{-1/2} f(x^0 t^{01/2}(-2b+1/t^0)^{1/2},\ (-2b+1/t^0)^{-1}),
\end{aligned}
$$

where $t^0 = e^{y^0}$.

If $g \in T_3$ is given by (1.1), we find

$$g = \exp(bj^+)\exp(cj^-)\exp(\tau j^3)$$

and

$$
\begin{aligned}
&T[(\exp bj^+)\ \exp(cj^-)(\exp \tau j^3)f](x,t)\\
&= [T(\exp bj^+)T(\exp cj^-)T(\exp \tau j^3)f](x,t)\\
&= (1-2bt)^{-1/2} exp[x-(-2b+1/t)^{1/2}(-2c+x^2t)^{1/2}\cdot \qquad (2.1)\\
&\cdot f((-2b+1/t)^{1/2}(-2c+x^2t)^{1/2},\ e^\tau(-2b+1/t)^{-1}).
\end{aligned}
$$

According to Miller [5. § 2.2], our realization of the representation $Q(1,m_0)$ of J_3 on the space generated by the function $f_m(x,y)$, $m \in S$ can be extended to a local representation T_3, where the group action is given by (2.1). The matrix elements of this local representation with respect to the basis f_m are uniquely determined by $Q(1,m_0)$, and we obtained the relations

$$
\begin{aligned}
&[T(g)f_{m_0+k}]\,(x,t) = \sum_{\ell=-\infty}^{\infty} A_{\ell k}(g)\, f_{m_0+k}(x,t)\\
&k = 0, \pm 1, \pm 2, \dots, \qquad (2.2)
\end{aligned}
$$

$$
\begin{aligned}
&t^{-1/2}(t^{-1}-2b)^{-1/2-m}\ \exp[m\tau + x - (t^{-1}-2b)^{1/2}(x^2t-2c)^{1/2}]\\
&S_m((t^{-1}-2b)^{1/2}\,(x^2t-2c)^{1/2})\\
&= \sum_{\ell=-\infty}^{\infty} A_{\ell,m-m_0}(g)\, S_{m_0+\ell} t^{m_0+\ell}, \qquad (2.3)
\end{aligned}
$$

where the matrix elements $A_{\ell k}(g)$ are given by [5, p. 56, (3.12)]

$$A_{\ell k}(g) = \frac{e^{(m_0+k)\tau}\,(c)^{(k-\ell+|k-\ell|)/2} b^{(\ell-k+|k-\ell|)/2}}{|k-\ell|!} \cdot {}_0F_1(|k-\ell|+1; bc) \tag{2.4}$$

valid for all the integral values of ℓ, k. Since $S_m(x), m \in C$ is analytic in x for all the nonzero values of x, the infinite series (2.3) converges absolutely for $|2bt/x| < 1$, $|2c/xt| < 1$. Thus our main generating function becomes

$$\begin{aligned}(-2bt+1)^{-m-1/2} & \exp[x - (-2bt+1)^{1/2}(-2c/t+x^2)^{1/2}) \\ & \cdot S_m((-2bt+1)^{1/2}(-2c/t+x^2)^{1/2}) \\ = \sum_{n=-\infty}^{\infty} & c^{(-n+|n|)/2}\, b^{(n+|n|)/2}\; {}_0F_1(|n|+1; bc) S_{m+n}(x) t^n.\end{aligned} \tag{2.5}$$

Several results of special functions are particular cases of the formula (2.5). If $c = 0$ and $b = 1$, equation (2.5) becomes

$$\begin{aligned}(1-2t)^{-m-1/2} & \exp[x(1-(1-2t)^{1/2})]\, S_m[x(1-2t)^{1/2}] \\ & = \sum_{n=0}^{\infty} S_{m+n}(x)\frac{t^n}{n!},\end{aligned} \tag{2.6}$$

where $m = 0, 1, 2, \ldots$. (2.6) is a known generating function [10, p. 256 (35)] and also follows from the known results [9, p. 78 (3.2); p. 79 (3.6)] by appealing the result

$$S_n(x) = (-1)^n n! G_n^{-1}(x, 1, 1, -2).$$

A special case of (2.6),

$$\sum_{n=0}^{\infty} S_n(x)\frac{t^n}{n!} = (1-2t)^{-1/2} \exp[x(1-(1-2t)^{1/2})]$$

can also be derived by suitably specializing a result due to Carlitz [1, p. 826 (8)].

If $b = 0$ and $t = 1$ in (2.5), we obtain

$$\exp[x - (x^2-2c)^{1/2}]\, S_m[(x^2-2c)^{1/2}]$$

$$(2.7)\qquad \sum_{n=0}^{\infty} \frac{c^n}{n!} S_{m-n}(x).$$

When $c = 0$, (2.5) yields a more familiar result [4, p. 50 (12)] for the simple Bessel polynomials $y_n(x)$

$$\sum_{n=0}^{\infty} y_{m+n}(x)\frac{t^n}{n!}$$

$$(2.8)\qquad = (1-2xt)^{-(m+1/2)} \exp[\frac{1-(1-2xt)^{1/2}}{x}]\, y_m(x(1-2xt)^{-1/2}).$$

If $bc \neq 0$, we can introduce the coordinates r, ν defined by $r = (ibc)^{1/2}$ and $\nu = (b/ic)^{1/2}$ such that $b = r\nu/2$, $c = -r/2\nu$. In this case, equation (2.5) yields the generating function

$$(1-r\nu t)^{-m-1/2} \exp[x - (1-r\nu t)^{1/2}(x^2 + r/\nu t)^{1/2}]$$

$$(2.9)\qquad (S_m((1-r\nu t)^{1/2}(x^2 - r/\nu t)^{1/2} = \sum_{n=-\infty}^{\infty} (-\nu)^n J_n(r) S_{m+n}(x) t^n,$$

where $J_n(r)$ is a Bessel function of integral order.

References

[1] Carlitz, L.: Some generating functions for Laguere polynomials, Duke Math. J. 35 (1968), 825-827.

[2] Erdelyi, A. et. al.: Tables of integral transforms, Vol. II, McGraw Hill, New York, London and Toronto, 1954.

[3] Krall, H. L., Frink, O.: A new class of orthogonal polynomials, The Bessel polynomials, Trans. Amer. Math. Soc. 65 (1949), 100-115.

[4] McBride, E. B.: Obtaining Generating Functions Springer-Verlag, New York, Heidelberg and Berlin, 1971.

[5] Miller, W.: Lie-theory and Special Functions, Academic Press, New York and London, 1968.

[6] Pontrjagin, L. S.: Topological Groups, Princeton Univ. Press, Princeton, New Jersey, 1958.

[7] Rainville, E. D.: Special Functions, Macmillan, New York, 1960.

[8] Schultz-Piszachich, W.: Eine Klasse isotroper Turbulenzfelder mit Permanenzeigenschaft und existenten Tripelkorrelationen, ZAMM 56 (1976), 303-314.

[9] Srivastava, H. M., Singhal, J. P.: A class of polynomials defined by generalized Rodrigues' formula, Ann. Mat. Pura Appl., IV. Ser. 90 (1971), 78-85.

[10] Srivastava, H. M.: Some orthogonal polynomials representing the energy spectral functions for a family of isotropic turbulence fields, ZAMM 64 (1984), 255-257.

[11] Werner, U., Pietzsch, W.: Die Polynome $S_n(x)$ zur Darstellung der Energiespektralfunktionen idealer turbulenzfelder ZAMM 58 (1978), 166-168.

REZIME

GENERIRAJUĆE FUNKCIJE ZA $S_N(X)$ SA TAČKE GLEDIŠTA LIOVE ALGEBRE

Generirajuće funkcije za polinome $S_N(X)$, uveo je nedavno W. Schultz – Piszachich [8] pri ispitivanju izotopne turbulencije polja, dobijene su korišćenjem reprezentacije Liovih grupa T_3 [5]. Nekoliko specijalnih slučajeva od interesa su takodje diskutovani.

Received by the editors February 13, 1987