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RECURRENCE RELATIONS FOR THE HERMITE SOLUTION OF AN ORDINARY DIFFERENTIAL EQUATION WITH POLYNOMIAL COEFFICIENTS

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Abstract

Systematic methods are presented for obtaining recurrence relations for coefficients in the Hermite series solution of linear differential equations with polynomial coefficients. The explicit form for the second order equation is given, and the numerical results illustrated by a fourth order initial value problem. The estimate of the error is obtained, as well.

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1. Introduction

In this paper we shall consider the differential equation

$$\sum_{r=0}^{l} p_r(x)y^{(r)}(x) = g(x) \tag{1}$$

with suitable boundary or initial conditions, where the functions $p_r(x)$ are either polynomials of a sufficiently low degree, or can be accurately approximated by such polynomials and we shall construct the approximate solution in the form of the Hermite series.

Such methods, known as spectral solutions, were recently developed from Fourier approximations, so that instead of using standard Fourier series, as in [3], the approximate solution is represented by some other orthogonal series. Most of the authors deal with the Chebyshev series, i.e: Paškovskij in [7], Morris and Horner in [5] and [4], Oliver in [6]. Only a few of them treat the other orthogonal series as well; Caunto in [1] uses Legendre polynomials and Chang and Wang in [2] the Hermite ones. In all these papers, the appropriate series solution is, most often, obtained by the collocation technique. Here, the direct method, proposed by Horner in [4], when Chebyshev series were used, will be carried out for the Hermite approximation. By means of some technical preliminaries we shall obtain the recurrence relation which can be readily applied to approximate the solution with extreme accuracy using only a small number of terms in the appropriate series. In this way we can automate the solution of equations of the given type, but care should be taken to investigate certain facts, such as the convergence of solution, singular points and the truncation point which determine the desired accuracy. The obtained recurrence relation, should also be investigated analytically.

2. The method of solution

Let $p_r(x)$, $r = 0,1,2$ denote the quadratic polynomials. Thus, the equation (1) for $l = 2$ becomes

$$\begin{aligned}&(c_1x^2 + c_2x + c_3)y''(x)+\\&+(c_4x^2 + c_5x + c_6)y'(x) + (c_7x^2 + c_8x + c_9)y(x) = g(x).\end{aligned} \tag{2}$$

Assume that the solution of (2) can be presented by an orthogonal series with respect to the Hermite orthogonal basis, in the form

$$y(x) = \sum_{k=0}^{\infty} a_k H_k(x). \tag{3}$$

It is well known that Hermite polynomials are generated by the differential equation

$$H_k''(x) - 2xH_k'(x) + 2kH_k(x) = 0$$

and that their explicit form can be obtained from the recurrence relation

$$\begin{aligned}H_{k+1}(x) - 2xH_k(x) + 2kH_{k-1}(x) = 0,\ \ k = 1,2,\dots,\\ H_0(x) = 1,\ H_1(x) = 2x.\end{aligned} \tag{4}$$

We, also, assume that $g(x)$ is a continuous function, so that it can, also, be expressed in the form of the Hermite series as

$$g(x) = \sum_{k=0}^{\infty} g_k H_k(x). \tag{5}$$

In order to obtain the recurrence relation for the evaluation of the coefficients a_k of the solution (3), we shall have to use the following results:

$$H_k'(x) = 2kH_{k-1}(x), \quad k = 1, 2, \ldots \tag{6}$$

$$H_k''(x) = 4k(k-1)H_{k-2}(x), \quad k = 2, 3, \ldots, \tag{7}$$

which give

$$y'(x) = 2\sum_{k=0}^{\infty}(k+1)a_{k+1}H_k(x) \tag{8}$$

and

$$y''(x) = 4\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}H_k(x). \tag{9}$$

For the construction of the approximate solution of the equation (2) we can state the following theorem:

Theorem 1. *The coefficients a_k in the solution (3) of equation (2) are determined by the system*

$$\sum_{\substack{i=k-2 \\ i \geq 0}}^{k+4} \left(\sum_{j=1}^{9} c_j \omega_{i,j}^{(k)} \right) a_i = 4g_k, \quad k = 0, 1, \ldots \tag{10}$$

where $\omega_{i,j}^{(k)}$ is the element of the i-th row and j-th column of the matrix $W^{(k)}$, whose non-zero elements are given in Table 1.

Proof. We shall first substitute (3), (5), (8) and (9) into (2) and then make use of the formulas

$$xH_k(x) = \frac{1}{2}(H_{k+1}(x) + 2kH_{k-1}(x)), \quad k = 1, 2, \ldots \tag{11}$$

and

$$x^2H_k(x) = \tfrac{1}{4}(H_{k+2}(x) + (4k+2)H_k(x) + 4k(k-1)H_{k-2}(x)), \quad k = 2, 3, \ldots \tag{12}$$

Hermite polynomials: The elements of matrix $W^{(k)}$

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9
$i=k-2, a_{k-2}$							1		
$i=k-1, a_{k-1}$				2(k-1)				2	
$i=k, a_k$	4k(k-1)				4k		2(2k+1)		4
$i=k+1, a_{k+1}$		8k(k+1)		4(2k+1)(k+1)		8(k+1)		4(k+1)	
$i=k+2, a_{k+2}$	8(2k+1)(k+1)(k+2)		16(k+1)(k+2)		8(k+1)(k+2)		4(k+1)(k+2)		
$i=k+3, a_{k+3}$		16(k+1)(k+2)(k+3)		8(k+1)(k+2)(k+3)					
$i=k+4, a_{k+4}$	16(k+1)(k+2)(k+3)(k+4)								

Table 1

After equating the coefficients of $H_k(x)$, $k = 0, 1, \ldots$ we come to the system

$$
\begin{aligned}
&c_7 a_{k-2} + (2c_4(k-1) + 2c_8)a_{k-1} + \\
&+(4c_1 k(k-1) + 4c_5 k + 2c_7(2k+1) + 4c_9)a_k + \\
&+(k+1)(8c_2 k + 4c_4(2k+1) + 8c_6 + 4c_8)a_{k+1} + \\
&+(k+2)(k+1)(8c_1(2k+1) + 16c_3 + 8c_5 + 4c_7)a_{k+2} + \\
&+(k+3)(k+2)(k+1)(16c_2 + 8c_4)a_{k+3} + \\
&+16(k+4)(k+3)(k+2)(k+1)c_1 a_{k+4} = 4g_k, \quad k = 2, 3, \ldots
\end{aligned}
\tag{13}
$$

(For $k = 1$ the first term is omitted, and for $k = 0$ the first two terms.)

The system (13), obviously, has the form (10), which proves the theorem.

□

Once the general form of the recurrence relation is known, a suitable truncation point in the series (3) is chosen, so that the solution is sufficiently well represented by the resulting n-th degree polynomial, i.e. the finite Hermite series. Then the system consisting of the appropriate equations from (10) and the equations representing the initial or boundary conditions is solved for $a_0, a_1, \dots, a_n$. For the evaluation of the finite Hermite series

$$y_n(x) = \sum_{k=0}^{n} a_k H_k(x) \tag{14}$$

the following algorithm can be constructed

$$\begin{aligned} &\text{let } f_{n+2} = f_{n+1} = 0 \\ &\text{let } f_{n-k} = a_{n-k} + 2x f_{n-k+1} - 2(n-k+1) f_{n-k+2} \\ &\qquad \text{for } k = 0, \dots, n \\ &\text{let } y_n = f_0. \end{aligned} \tag{15}$$

3. The generalization

The differential equations of a higher order with the second degree polynomial coefficients can be treated in the same manner. For the general case (1), which can be written in the form

$$\sum_{r=0}^{l} \left(\sum_{j=0}^{2} c_{r,j} x^j \right) y^{(r)}(x) = g(x), \tag{16}$$

we can state the following theorem:

Theorem 2. *The system for evaluating the coefficients a_k, $k = 0, 1, \dots$ of the Hermite approximate solution (3) of equation (16) has the form*

$$\sum_{\substack{i = k-2 \\ i \geq 0}}^{k+l+2} \left(\sum_{r=0}^{l} \sum_{j=0}^{2} c_{r,j} \omega_{i,r,j}^{(k)} \right) a_i = g_k \quad k = 0, 1, \dots \tag{17}$$

Proof. Starting from (16) we come to the equations

$$\sum_{r=0}^{l} 2^r \Big(\frac{(k+r-2)!}{4(k-2)!} c_{r,2} a_{k+r-2} + \frac{(k+r-1)!}{2(k-1)!} c_{r,1} a_{k+r-1} +$$

$$(18) \qquad + \frac{(k+r)!}{k!} (c_{r,0} + \frac{2k+1}{2} c_{r,2}) a_{k+r} + \frac{(k+r+1)!}{k!} c_{r,1} a_{k+r+1} +$$

$$+ \frac{(k+r+2)!}{k!} c_{r,2} a_{k+r+2} \Big) = g_k \quad k = 0, 1, \ldots$$

which are obtained by using the same technique as in the proof of Theorem 1, only that the generalized formula

$$(19) \qquad y^{(r)}(x) = 2^r \sum_{k=0}^{\infty} \frac{(k+r)!}{k!} a_{k+r} H_k(x) \quad r = 0, 1, \ldots,$$

is used instead of (8) and (9). It is obvious that (18) can be written down in the form (17) and, thus, the proof is comleted.

□

The explicit form for the elements $\omega_{i,r,j}^{(k)}$ of $W_l^{(k)}$ can be easily found for a fixed l.

4. The error estimate

When we ask for the approximate solution (14) of the problem described by the differential equation (16) and l boundary or initial conditions, we have to determine the coefficients a_k from the system

$$(20) \qquad \sum_{j=0}^{n} A_{k,j} a_j = \tilde{g}_k \quad k = 0, 1, \ldots, n$$

where the first $n-l+1$ equations are of the form (17) and the last l equations represent boundary or initial conditions. Thus, the system of $n+1$ linear algebric equations is solved instead of the infinite one

$$(21) \qquad \sum_{j=0}^{\infty} A_{k,j} \tilde{a}_j = \tilde{g}_k \quad k = 0, 1, \ldots$$

which determines the coefficients of the exact solution (3), which are now marked as $\tilde{a}_k$ to distingwish them from the coefficients in (14).

Our aim is to estimate the error

$$r(x) = |y_n(x) - y(x)|, \tag{22}$$

where we have

$$y_n(x) - y(x) = \sum_{k=0}^{n}(a_k - \tilde{a}_k)H_k(x) - \sum_{k=n+1}^{\infty} \tilde{a}_k H_k(x). \tag{23}$$

We can prove the following theorem, using the idea which Oliver developed in [6] for the Chebyshev approximation.

Theorem 3. *The error (22) for the problem described by (16) and the l suitable boundary or initial conditions is of the form*

$$r(x) = |\sum_{k=n+1}^{\infty} r_k(x)\tilde{a}_k|. \tag{24}$$

Proof. Substracting (21) from (20), we get

$$\sum_{j=0}^{n} A_{k,j}(a_j - \tilde{a}_j) = \sum_{i=n+1}^{\infty} A_{k,i}\tilde{a}_i, \quad k = 0, 1, \ldots \tag{25}$$

Further, we can define the numbers $q_{j,i}$, $j = 0, 1, \ldots, n$, $i = n+1, \ldots$ such that they represent the solution of the system

$$\sum_{j=0}^{n} A_{k,j} q_{j,i} = A_{k,i}, \quad k = 0, 1, \ldots \tag{26}$$

From (25), now, we get

$$a_j - \tilde{a}_j = \sum_{i=n+1}^{\infty} q_{j,i}\tilde{a}_i, \tag{27}$$

which, used in (23), gives (24) with

$$r_i(x) = \sum_{j=0}^{n} q_{j,i} H_j(x) - H_i(x). \tag{28}$$

□

For the practical use of the error estimate (24) the following requirements must be fulfilled : the summation must be dominated by the first few terms, which is indicated by the decrease of $a_{n+1}, a_{n+2}, \ldots$, and some estimate for the order of magnitude of $\tilde{a}_i$, $i = n+1, \ldots$ must be available. For these estimates we can use the values of a_i obtained for larger n.

5. The numerical example

As the numerical example we shall construct the approximate solution in the form (14) for the initial value problem due to Oliver [6].

(29) $$y^{(4)}x - 401y''(x) + 400y(x) = -1 + 200x^2$$

(30) $$y(0) = y'(0) = y''(0) = y'''(0) = 1,$$

with the exact solution

(31) $$y(x) = 1 + \frac{x^2}{2} + shx.$$

With respect to the elements of $W_4^{(k)}$, the system obtained from (20) is

(32) $$\begin{array}{ll} 400a_k - 1604(k+1)(k+2)a_{k+2} + & \\ 16(k+1)(k+2)(k+3)(k+4)a_{k+4} = g_k, & k = 0,\dots,n-4 \\ g_0 = 99, g_1 = 0, g_2 = 50, g_k = 0, & k = 3,\dots,n-4, \end{array}$$

and the initial conditions give

$$\sum_{k=0}^{n} a_k H_k(0) = 1, \quad 2\sum_{k=1}^{n} k a_k H_{k-1}(0) = 1,$$

(33) $$4\sum_{k=2}^{n} k(k-1)a_k H_{k-2}(0) = 1 \quad \text{and}$$

$$8\sum_{k=3}^{n} k(k-1)(k-2)a_k H_{k-3}(0) = 1,$$

with respect to

(34) $$H_{2s}(0) = (-1)^s \frac{(2s)!}{s!}, \quad H_{2s+1}(0) = 0, \quad s = 0,1,\dots$$

The values for $a_k, k = 0,\dots,n$ are presented in Table 2 for $n = 5,7,9$. It is obvious that $n = 5$ is the smallest possible degree because of the form of the system (23),(24), and the rapid decrease of the coefficients a_k disables the increase of the accuracy when n takes greater values than the presented ones.

In Table 3 the error

(35) $$d(x) = |y_n(x) - y(x)|$$

and the error estimate -

$$r(x) = |r_{n+1}(x)a_{n+1}| \tag{36}$$

are evaluated in the points

$$x_i = \frac{i}{10}, \quad i = 0, 1, \ldots, 10 \tag{37}$$

together with the appropriate values of the exact solution $y(x)$. It can be easily seen that the error estimate (31) is sufficient, though, when approaching zero, the approximate solution is more accurate than we might expect referring to (36).

a_k	$n = 5$	$n = 7$	$n = 9$
a_0	1.25	1.25	1.25
a_1	0.643	0.642	0.642
a_2	0.125	0.125	0.125
a_3	$2.7 * 10^{-2}$	$2.7 * 10^{-2}$	$2.7 * 10^{-2}$
a_4	0	0	0
a_5	$3 * 10^{-4}$	$3.3 * 10^{-4}$	$3.3 * 10^{-4}$
a_6	0	0	0
a_7	$1.8 * 10^{-6}$	$2 * 10^{-6}$	0
a_8		0	0
a_9		$6.5 * 10^{-9}$	0

Table 2

		$n=5$		$n=7$	
x	$y(x)$	$d(x)$	$r(x)$	$d(x)$	$r(x)$
0	1	0	$8*10^{-14}$	0	$6*10^{-16}$
0.1	1.105	$1.2*10^{-8}$	$7.7*10^{-8}$	$9.3*10^{-10}$	$5.4*10^{-8}$
0.2	1.221	$3.7*10^{-7}$	$3.1*10^{-5}$	$3.2*10^{-8}$	$2.1*10^{-7}$
0.3	1.349	$2.8*10^{-6}$	$6.8*10^{-5}$	$2.4*10^{-7}$	$4.6*10^{-7}$
0.4	1.491	$1.2*10^{-5}$	$1.2*10^{-4}$	$9.7*10^{-7}$	$7.8*10^{-7}$
0.5	1.646	$3.5*10^{-5}$	$1.8*10^{-4}$	$2.9*10^{-6}$	$1.2*10^{-6}$
0.6	1.817	$8.6*10^{-5}$	$2.5*10^{-4}$	$6.8*10^{-6}$	$1.5*10^{-6}$
0.7	2.004	$1.8*10^{-4}$	$3.2*10^{-4}$	$1.4*10^{-5}$	$1.9*10^{-6}$
0.8	2.208	$3.4*10^{-4}$	$4.0*10^{-4}$	$2.6*10^{-5}$	$2.2*10^{-6}$
0.9	2.432	$6.0*10^{-4}$	$4.7*10^{-4}$	$4.3*10^{-5}$	$2.5*10^{-6}$
1.0	2.675	$9.8*10^{-4}$	$5.3*10^{-4}$	$6.7*10^{-5}$	$2.6*10^{-6}$

		$n=9$	
x	$y(x)$	$d(x)$	$r(x)$
0	1	0	$2*10^{-16}$
0.1	1.105	0	$2.8*10^{-8}$
0.2	1.221	$1.9*10^{-9}$	$1.1*10^{-7}$
0.3	1.349	$1.5*10^{-8}$	$2.3*10^{-7}$
0.4	1.491	$6.0*10^{-8}$	$3.8*10^{-7}$
0.5	1.646	$1.7*10^{-7}$	$5.5*10^{-7}$
0.6	1.817	$4.0*10^{-7}$	$7.0*10^{-7}$
0.7	2.004	$8.0*10^{-7}$	$8.2*10^{-7}$
0.8	2.208	$1.4*10^{-6}$	$9.0*10^{-7}$
0.9	2.432	$2.3*10^{-6}$	$9.1*10^{-7}$
1.0	2.675	$3.4*10^{-6}$	$8.6*10^{-7}$

Table 3

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REZIME

REKURENTNE RELACIJE ZA HERMITOVO REŠENJE OBIČNIH DIFERENCIJALNIH JEDNAČINA SA POLINOMNIM KOEFICIJENTIMA

Prikazan je sistem za dobijanje rekurentnih relacija za nalaženje približnog rešenja linearnih diferencijalnih jednačina sa polinomnim koeficijentima u obliku Hermitovog reda. Date su eksplicitne forme za jednačine drugog reda, a numerički rezultati ilustruju početni problem četvrtog reda. Takodje je dobijena i ocena greške.

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