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SOME THEOREMS ABOUT CONSTRUCTIONS OF a-COIDEALS ON THE CARTESIAN PRODUCT OF SETS

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Abstract

In constructive mathematics determining of a-coideals of the Cartesian product of sets is heavier from determining of filters in classical mathematics. This short note contains some theorems about constructions of a-coideals of the Cartesian product of sets.

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For all notions, which we use here, we referee to the books [1], [2], [6] and to the papers [3], [4], [5]. In the papers [3] and [4] the author defined the notion of a-coideal of sets. The paper [5] contains two theorems about constructions of a-coideals on the Cartesian product $X \times Y$ of sets X and Y by using a-coideals of sets X and Y.In this paper it is shown the construction of a-coideal of Cartesian product $\prod_{t\in T} X_t$ where T is not subfinite set.

A set is an ordered triples $(X, =, \neq)$, where $=$ is an equivalence relation and where $\neq$ is an inequality relation. A subset Y of X is **inhabited** if and only if $(\exists x \in X)(x \in Y)$. The **empty set** $\emptyset$ is in the set which cannot be inhabited. We write $x\#Y$ if and only if $(\forall y \in Y)(y \neq x)$. In the power - set $P(X)$ of X we define $Y = Z$ if and only if $Y \subseteq Z$ and $Z \subseteq Y$,and $Y \neq Z$ if and only if $(\exists y \in Y)(y\#Z)$ or $(\exists z \in Z)(z\#Y)$. A family of $P(X)$, indexed by a set $(T,=,\neq)$, is a total function $f : T \to P(X)$. A family K of $P(X)$ will be called an **a-coideal** if and only if

$$X \in K,\ Z_1 \subseteq Z_2 \wedge Z_1 \in K \Rightarrow Z_2 \in K, Z_1 \cup Z_2 \in K \Rightarrow Z_1 \in K \vee Z_2 \in K.$$

The a-coideal K is **strong** if and only if $\emptyset\#K$. The family $B \subseteq P(X)$ is called a **basis** of an a-coideal of a set X if and only if

$$\emptyset\#B,\ (\forall S, V \in P(X))(S \cup V \in B \Rightarrow S \in B \wedge V \in B).$$

If B is a basis of an a-coideal of a set X, then the family $\{Z \in X : (\exists S \in B)(S \subseteq Z)\}$ is the a-coideal of X induced by B.

We first recall the theorem about a construction of an a-coideal of the Cartesian product $\prod_{i=1}^{n} X_i$ of sets X_i $(i = 1, ..., n)$, which is a generalization of result of Theorem 5 from the paper [5]. Because of that we need the following notions:

$$E \subseteq \prod_{i=1}^{n} X_i,$$

$$x_i \in X_i\ (i = 1, ..., n),$$

$$E^j_{x_1 ... x_{j-1} x_{j+1} ... x_n}$$

$$\equiv \{t \in X_j : (x_1, ..., x_{j-1}, t, x_{j+1}, ..., x_n) \in E\}\ (j = 1, ..., n)$$

and lemmas:

Lemma 1.

$$(E_1 \cup E_2)^j_{x_1 ... x_{j-1} x_{j+1} ... x_n} = (E_1)^j_{x_1 ... x_{j-1} x_{j+1} ... x_n} \cup (E_2)^j_{x_1 ... x_{j-1} x_{j+1} ... x_n}.$$

Proof.

$$t \in (E_1 \cup E_2)^j_{x_1 ... x_{j-1} x_{j+1} ... x_n}$$

$$\Leftrightarrow (x_1, ..., x_{j-1}, t, x_{j+1}, ..., x_n) \in (E_1 \cup E_2)$$

$$\Leftrightarrow (x_1, ..., x_{j-1}, t, x_{j+1}, ..., x_n) \in E_1 \vee (x_1, ..., x_{j-1}, t, x_{j+1}, ..., x_n) \in E_2$$

$$\Leftrightarrow t \in (E_1)^j_{x_1 ... x_{j-1} x_{j+1} ... x_n} \vee t \in (E_2)^j_{x_1 ... x_{j-1} x_{j+1} ... x_n}$$

$$\Leftrightarrow t \in (E_1)^j_{x_1 ... x_{j-1} x_{j+1} ... x_n} \cup (E_2)^j_{x_1 ... x_{j-1} x_{j+1} ... x_n}. \square$$

Lemma 2.

$$\{t \in X_k : (E_1)^j_{x_1 \dots x_{j-1} x_{j+1} \dots x_n} \cup (E_2)^j_{x_1 \dots x_{j-1} x_{j+1} \dots x_n} \in K_k\}$$
$$\subseteq \{t \in X_k : (E_1)^j_{x_1 \dots x_{j-1} x_{j+1} \dots x_n} \in K_k\}$$
$$\cup \{t \in X_k : (E_2)^j_{x_1 \dots x_{j-1} x_{j+1} \dots x_n} \in K_k\} k \in \{1, \dots, n\} \setminus \{j\}.$$

Proof.

$$x_k \in \{t \in X_k : (E_1)^j_{x_1 \dots x_{j-1} x_{j+1} \dots x_n} \cup (E_2)^j_{x_1 \dots x_{j-1} x_{j+1} \dots x_n} \in K_k\}$$
$$\Leftrightarrow (E_1)^j_{x_1 \dots x_{j-1} x_{j+1} \dots x_n} \cup (E_2)^j_{x_1 \dots x_{j-1} x_{j+1} \dots x_n} \in K_k$$
$$\Leftrightarrow (E_1)^j_{x_1 \dots x_{j-1} x_{j+1} \dots x_n} \in K_k \vee (E_2)^j_{x_1 \dots x_{j-1} x_{j+1} \dots x_n} \in K_k$$
$$\Leftrightarrow x_k \in \{t \in X_k : (E_1)^j_{x_1 \dots x_{j-1} x_{j+1} \dots x_n} \in K_k\}$$
$$\cup \{t \in X_k : (E_2)^j_{x_1 \dots x_{j-1} x_{j+1} \dots x_n} \in K_k\}. \square$$

Lemma 3. *Let* $< K_i >_{i \in I}$ *be a family of a-coideals of a set* X. *Then the* $\bigcup_{i \in I} K_i$ *is an a-coideal of the set* X.

Proof. Routine.□

Theorem 1. *Let* K_i $(i = 1, \dots, n)$ *be an a-coideal of a set* X_i $(i = 1, \dots, n)$ *respectively, and let* s *be a permutation of the set* $\{1, \dots, n\}$. *Then the family*

$$K_s = \{E \subseteq \prod_{i=1}^{n} X_i \{x_{s(1)} \in X_{s(1)} : \{x_{s(2)} \in X_{s(2)} : \dots \{x_{s(n-1)} \in X_{s(n-1)} :$$
$$E^{s(n)}_{x_{s(1)} \dots x_{s(n-1)}} \in K_{s(n)}\} \in K_{s(n-1)}\} \dots \in K_{s(2)}\} \in K_{s(1)}\}$$

is an a-coideal of the set $\prod_{i=1}^{n} X_i$.

Proof.

(i) $E \in K_s$. Then
$$E^{s(n)}_{x_{s(1)} \dots x_{s(n-1)}} \neq \emptyset$$
because $K_i \# \emptyset$ (for all $i \in \{1, \dots, n\}$). Therefore $E \neq \emptyset$.

(ii) As

$$(\prod_{i=1}^{n} X_i)^{s(n)}_{x_{s(1)}\dots x_{s(n-1)}} = X_{s(n)} \ (\forall s \in S_n)$$

and $X_{s(n)} \in K_{s(n)}$ we have

$$(\prod_{i=1}^{n} X_i)^{s(n)}_{x_{s(1)}\dots x_{s(n-1)}} \in K_{s(n)}.$$

In this way, we have

$$\{x_{s(n-1)} \in X_{s(n-1)} : (\prod_{i=1}^{n} X_i)^{s(n)}_{x_{s(1)}\dots x_{s(n-1)}} \in K_{s(n)}\} = X_{s(n-1)}$$

and

$$\{x_{s(1)} \in X_{s(1)} : \{\dots(\prod_{i=1}^{n} X_i)^{s(n)}_{x_{s(1)}\dots x_{s(n-1)}} \in K_{s(n)}\}\dots\} = X_{s(1)}.$$

So $\prod_{i=1}^{n} X_i \in K_s$.

(iii) Let E_1, E_2 be arbitrary elements of $P(\prod_{i=1}^{n} X_i)$ such that $E_1 \subseteq E_2$ and $E_1 \in K_s$. Then

$$(E_1)^{s(n)}_{x_{s(1)}\dots x_{s(n-1)}} \subseteq (E_2)^{s(n)}_{x_{s(1)}\dots x_{s(n-1)}}$$

and

$$\{x_{s(n-1)} \in X_{s(n-1)} : (E_2)^{s(n)}_{x_{s(1)}\dots x_{s(n-1)}} \in K_{s(n)}\} \in K_{s(n-1)}$$

because

$$\{x_{s(n-1)} \in X_{s(n-1)} : (E_1)^{s(n)}_{x_{s(1)}\dots x_{s(n-1)}} \in K_{s(n)}\} \in K_{s(n-1)}$$

and etc. Thus $E_2 \in K_s$.

(iv) Suppose that E_1, E_2 are elements of $P(\prod_{i=1}^{n} X_i)$ such that $E_1 \cup E_2 \in K_s$. Then

$$\{x_{s(1)} \in X_{s(1)} : \dots\{x_{s(n-1)} \in X_{s(n-1)} :$$

$$(E_1 \cup E_2)^{s(n)}_{x_{s(1)}\dots x_{s(n-1)}} \in K_{s(n)}\} \in K_{s(n-1)}\} \dots\} \in K_{s(1)}$$

$$\Leftrightarrow \{x_{s(1)} \in X_{s(1)} : \dots\{x_{s(n-1)} \in X_{s(n-1)} :$$

$$(E_1)^{s(n)}_{x_{s(1)}\dots x_{s(n-1)}} \cup (E_2)^{s(n)}_{x_{s(1)}\dots x_{s(n-1)}} \in K_{s(n)}\} \in K_{s(n-1)}\} \dots\} \in K_{s(1)}$$

$$\Rightarrow \{x_{s(1)} \in X_{s(1)} : \dots\{x_{s(n-1)} \in X_{s(n-1)} : (E_1)^{s(n)}_{x_{s(1)}\dots x_{s(n-1)}} \in K_{s(n)}\}$$

$$\cup\{x_{s(n-1)} \in X_{s(n-1)} : (E_2)^{s(n)}_{x_{s(1)}\dots x_{s(n-1)}} \in K_{s(n)}\}$$

$$\in K_{s(n-1)}\} \dots\} \in K_{s(1)}$$

$$\Rightarrow \dots$$

$$\Rightarrow \{x_{s(1)} \in X_{s(1)} :$$

$$\dots\{x_{s(n-1)} \in X_{s(n-1)} : (E_1)^{s(n)}_{x_{s(1)}\dots x_{s(n-1)}} \in K_{s(n)}\}\dots\} \in K_{s(1)}$$

$$\vee\{x_{s(1)} \in X_{s(1)} : \dots\{x_{s(n-1)} \in X_{s(n-1)} :$$

$$(E_2)^{s(n)}_{x_{s(1)}\dots x_{s(n-1)}} \in K_{s(n)}\}\dots\} \in K_{s(1)}$$

$$\Leftrightarrow E_1 \in K_s \vee E_2 \in K_s. \square$$

Corollary 1. *Let L be a subset of the set S_n. Then the family $\bigcup_{s\in L} K_s$ is an a-coideal of the set $\prod_{i=1}^{n} X_i$.*

The question of generalization of Theorem 1 on the Cartesian product $\prod_{t\in T} X_t$, where T is not subfinite set, is open. Now, we give two theorems about construction of a-coideals of the Cartesian product $\prod_{t\in T} X_t$.

Theorem 2. *Let $< X_t >_{t\in T}$ be a family of sets and let K_t $(t \in T)$ be an a-coideal of the set X_t $(t \in T)$ respectively, and let K_0 be an a-coideal of the set T. Then the family B, defined by*

$$E \in B \Leftrightarrow \{t \in T : \{f(t) \in X_t : f \in E\} \in K_t\} \in K_0$$

is the basis of a-coideal of the set $\prod_{t\in T} X_t$.

Proof.

(i)

$$E \in B \Leftrightarrow \{t \in T : \{f(t) \in X_t : f \in E\} \in K_t\} \in K_0 \# \emptyset$$

$$\Rightarrow (\exists t \in T)(f(t) \in X_t : f \in E\} \in K_t \# \emptyset)$$

$$\Rightarrow E \neq \emptyset;$$

(ii)

$$(E_1 \cup E_2) \in B \Leftrightarrow \{t \in T : \{f(t) \in X_t : f \in E_1 \cup E_2\} \in K_t\} \in K_0$$

$$\Rightarrow \{t \in T : \{f(t) \in X_t : f \in E_1\} \in K_t\}$$

$$\cup \{t \in T : \{f(t) \in X_t : f \in E_2\} \in K_t\} \in K_0$$

$$\Rightarrow \{t \in T : \{f(t) \in X_t : f \in E_1\} \in K_t\} \in K_0$$

$$\vee \{t \in T : \{f(t) \in X_t : f \in E_2\} \in K_t\} \in K_0$$

$$\Leftrightarrow E_1 \in B \vee E_2 \in B$$

because

$$g(t) \in \{f(t) \in X_t : f \in E_1 \cup E_2\} \Leftrightarrow g \in E_1 \cup E_2$$

$$\Leftrightarrow g \in E_1 \vee g \in E_2$$

$$\Leftrightarrow g(t) \in \{f(t) \in X_t : f \in E_1\} \vee g(t) \in \{f(t) \in X_t : f \in E_2\}$$

$$\Leftrightarrow g(t) \in \{f(t) \in X_t : f \in E_1\} \cup \{f(t) \in X_t : f \in E_2\}$$

and

$$s \in \{t \in T : \{f(t) \in X_t : f \in E_1 \cup E_2\} \in K_t\}$$

$$\Leftrightarrow \{f(s) \in X_s : f \in E_1 \cup E_2\} \in K_s$$

$$\Leftrightarrow \{g(s) \in X_s : g \in E_1\} \cup \{h(s) \in X_s : h \in E_2\} \in K_s$$

$$\Leftrightarrow \{g(s) \in X_s : g \in E_1\} \in K_s \vee \{h(s) \in X_s : h \in E_2\} \in K_s$$

$$\Leftrightarrow s \in \{t \in T : \{g(t) \in X_t : g \in E_1\} \in K_t\}$$

$$\vee s \in \{t \in T : \{h(t) \in X_t : h \in E_2\} \in K_t\}$$

$$\Leftrightarrow s \in \{t \in T : \{g(t) \in X_t : g \in E_1\} \in K_t\}$$

$$\cup \{t \in T : \{h(t) \in X_t : h \in E_2\} \in K_t\}. \square$$

Corollary 2. *Let $< X_t >_{t \in T}$ be a family of sets and for $t \in T$ let K_t be an a-coideal of the set X_t $(t \in T)$. Then the family D, defined by*

$$E \in D \Leftrightarrow (\exists t \in T)(\{f(t) \in X_t;\ f \in E\} \in K_t)$$

is the basis of an a-coideal of the set $\prod_{t \in T} X_t$.

Proof. In the Theorem 2 take $K_0 = \{V \subseteq T : V \neq \emptyset\}$.□

Theorem 3. *Let* $< X_t >_{t \in T}$ *be a family of sets and for* $t \in T$ *let* K_t *be an a-coideal of the set* X_t $(t \in T)$. *Then the family* B_t $(t \in T)$, *defined by*

$$E \in B_t \Leftrightarrow \mathrm{proj}_t E \in K_t$$

is the basis of an a-coideal of the set $\prod_{t \in T} X_t$.

Proof.

(i)

$$E \in B_t \Leftrightarrow \mathrm{proj}_t E \in K_t \# \emptyset$$
$$\Rightarrow \mathrm{proj}_t E \neq \emptyset$$
$$\Rightarrow E \neq \emptyset;$$

(ii)

$$E_1 \cup E_2 \in B_t \Leftrightarrow \mathrm{proj}_t(E_1 \cup E_2) \in K_t$$
$$\Leftrightarrow \mathrm{proj}_t E_1 \cup \mathrm{proj}_t E_2 \in K_t$$
$$\Rightarrow \mathrm{proj}_t E_1 \in K_t \vee \mathrm{proj}_t E_2 \in K_t$$
$$\Leftrightarrow E_1 \in B_t \vee E_2 \in B_t.$$ □

Corollary 3. *Let* L *be a subset of the set* T. *Then the family* $\bigcup_{t \in L} B_t$ *is the basis of an a-coideal of the set* $\prod_{t \in T} X_t$. *Specially* $\bigcup_{t \in T} B_t = D$.

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REZIME

NEKE TEOREME O KONSTRUKCIJI a-KOIDEALA NA KARTEZIJEVOM PROIZVODU SKUPOVA

U konstruktivnoj matematici je određivanje a-koideala Kartezijevog proizvoda skupova teže od određivanja filtera u klasičnoj matematici. Ova kratka beleška sadrži neke teoreme o konstrukciji a-koideala Kartezijevog proizvoda skupova.

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