Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 22, 1 (1992), 107-110

Review of Research
Faculty of Science
Mathematics Series

TWO COMMUTATIVITY RESULTS FOR RINGS

Mohd. Ashraf, Murtaza A. Quadri
Department of Mathematics, Aligarh Muslim University
Aligarh - 202001 - India

Abstract

In the paper two commutativity theorems are proved: (1) If R is a semi prime ring and $n > 1$ a fixed positive integer such that either $[[x,y]^n - [x^n, y^n], x] = 0$ or $[(x \circ y)^n - (x^n \circ y^n), x] = 0$, then R is commutative (ii) If R is a ring in which for each x, y in R there exists a positive integer $n = n(x, y) > 1$ such that $(xy)^n = yx$, then R is commutative.

AMS Mathematics Subject Classification (1991): 16R10
Key words and phrases: Commutativity, rings.

Let R be an associative ring with a center $Z(R)$. For any x, y in R as usual, $[x, y] = xy - yx$ and $(x \circ y) = xy + yx$, the well-known Lie and Jordan products, respectively. In a paper, the authors together with M. A. Khan [6] replaced the associative product of the ring R by the above defined non-associative products in the identity $(xy)^2 = x^2y^2$ (c.f. Johnsen, Outcalt and Yagub, Amer. Math. Monthly 74 (1968)) and investigated the commutativity of associative structure. The result to whi ch we refer is as follows: "If R is 2-torsion free ring with unity 1 in which either $[x, y]^2 = [x^2, y^2]$ or $(x \circ y)^2 = x^2 \circ y^2$, then R is commutative". Now, we shall extend this study further and prove the following:

Theorem 1. *Let R be a semi prime ring and $n > 1$ be a fixed positive integer. If R satisfies any one of the following polynomial identities:*

(1) $$[[x,y]^n - [x^n, y^n], x] = 0 \text{ for all } x, y \text{ in } R,$$

(2) $$[[x,y]^n - [x^n, y^n], y] = 0 \text{ for all } x, y \text{ in } R,$$

(3) $$[[x \circ y]^n - [x^n \circ y^n], x] = 0 \text{ for all } x, y \text{ in } R,$$

(4) $$[[x \circ y]^n - [x^n \circ y^n], y] = 0 \text{ for all } x, y \text{ in } R,$$

then in each case R must be commutative.

We notice that a Boolean ring (satisfying $x^2 = x$) is necessarily commutative and so in such rings $(xy)^2 = xy$ and $(xy)^2 = yx$. But there exist non-Boolean rings satisfying $(xy)^2 = xy$ or $(xy)^2 = yx$. Very recently Sercoid and MacHale [7] have studied the commutativity of rings with $(xy)^2 = xy$ and $(xy)^{n(x,y)} = xy$. In [6] the commutativity of rings satisfying $(xy)^2 = yx$ has been investigated. Now, we can generalize the mentioned result [6, Theorem 3] as follows:

Theorem 2. *Let R be a ring in which $(xy)^n = yx$ for all x, y in R, where $n = n(x,y) > 1$ is an integer. Then R is commutative.*

Proof of Theorem 1. If R is a semi prime ring satisfying the hypothesis of the theorem, then it is isomorphic to a subdirectsum of prime rings R_i each of which as a homomorphic image of R satisfies the hypothesis placed on R. Thus, we may assume that the ring R is prime satisfying any one of the polynomial identities (1) – (4). By Posner's theorem [2, Page 465], the central quotient of R is a central simple algebra over a field.

Case I
Let the ground field be finite. Then, the center $Z(R)$ of R is a finite integral domain and R is equal to its central quotient of R. Hence R is a matrix ring $M_r(K)$ for some $r \geq 1$ and some field K.

Case II
Let the ground field be infinite and $P(x,y) = 0$ is the polynomial identity for R. We write $P = P_0 + P_1 + \ldots + P_m$, where P_i is the homogeneous polynomial in x, y. Then $P_0 = P_1 = \ldots = P_m = 0$ for every x, y in R, since the center of R is infinite. Thus $P_0 = P_1 = \ldots = P_m = 0$ is also valid in the central

quotient of R. Thus $P = P_0 + P_1 + \ldots + P_m = 0$ is satisfied by elements in $A \underset{K}{\otimes} L$ where A is a central quotient of $R, K = Z(A)$, L any field extension of K. In particular takin L to be a splitting field of A, $A \underset{K}{\otimes} L = M_r(L)$. Thus $P = 0$ is satisfied by elements in $M_r(L)$.

Now, we claim that $r = 1$ in every case. Let e_{ij} $1 \leq i, j \leq r$, be the matrix in $M_2(K)$ with 1 on the position (i, j) and with zeros elsewhere.

(i) If R satisfies (1) or (2), then $P(e_{11}, e_{11} + e_{12}) \neq 0$.

(ii) If R satisfies (3) or (4), then again $P(e_{11}, e_{12} + e_{22}) \neq 0$.

Thus in every case we get a contradiction and hence $r = 1$. Now since $r = 1$, the central quotient is contained in the respective ground field and R itself is commutative.

The ring of 3×3 strictly upper triangular matrices over a ring provides an example to show that the above theorem is not valid for arbitrary rings.

A ring R without a proper nil ideal is necessarily semiprime. Hence the following Corollary is a special case of our theorem. But it may be somewhat interesting to give an easy direct proof of the same.

Corollary 1. *Let R be a ring without proper nil ideals and $n > 1$ be a fixed positive integer. In R satisfies one of the polynomial identities (1) – (4), then R must be commutative.*

Proof. Each of the conditions (1) – (4) is a polynomial identity $P(x, y) = 0$, where $P(x, y)$ is a polynomial in two noncommutative variables with rational integral coefficients at least one of which is equal to 1. Moreover, none of the rings $M_2(GF(p))$ (p a prime) satisfies this identity. In fact, for $e_{11} \epsilon M_2(GF(p))$ we have $P(e_{11}, e_{11} + e_{12}) \neq 0$ in the case (1) or (2), and $P(e_{11}, e_{12} + e_{22}) \neq 0$ in the case (3) or (4). Hence, the ring R must be commutative according to a well known result due to T. Kezlan [1, Theorem 1].

Proof of Theorem 2. R satisfies the condition $(xy)^n = yx$ for all x, y in R and $n = n(x, y) > 1$. Clearly $xy = 0$ implies $yx = 0$. It follows that for any nilpotent element x in R, (xy) is nilpotent for all y in R. Thus the nilpotent elements of R annihilate R on both sides, and are therefore, central. Now for x in R, there exists $m = m(x)$ such that $x^{2m} = x^2$ and $2m \neq 2$. Thus R is a periodic ring with central nilpotent elements. Hence, commutativity of ring R follows by the theorem of Herstein [3].

Acknowledgement: The authors are thankful to the referee for his valuable comments specially for supplying the proof of the Corollary.

References

[1] Bell, H.E.: On some commutativity theorems of Herstein, Arch. Math, 24(1973), 34-38.

[2] Cohn, P.M.: Algebra, Vol. 2 John Wiley and Sons, Inc. London, 1977.

[3] Herstein, I.N.: A note on rings with central nilpotent elements, Proc. Amer. Math. Soc., 5(1954), 620.

[4] Jacobson, N.: Structure of rings, Amer. Math. Soc. Colloq. Publ., 37, Providence, 1956.

[5] Quadri, M.A., Ashraf, Mohd.: Two commutativity theorems for semi prime rings, Indian J. Pure Appl. Math., 17(1986), 1207-1209.

[6] Quadri,M.A., Khan, M.A., Ashraf, Mohd.: Some elementary Commutativity theorems for rings, The Math. Student 56(1988), 223-226.

[7] Ó Sercóid, M., MacHale, D.: Two elementary generalizations of Boolean rings, Amer. Math. Monthly 93(1986), 121-122.

REZIME

DVA REZULTATA O KOMUTATIVNOSTI ZA PRSTENE

U radu su dokazane dve teoreme o komutativnosti: (i) Ako je R semi prost prsten i $n > 1$ fiksiran ceo broj takav da je ili $[[x,y]^n - [x^n,y^n],x] = 0$ ili $[(x \circ y)^n - (x^n \circ y^n),x] = 0$, tada je R komutativan, (ii) ako je R prsten u kojem za svako $x, y \in R$ postoji pozitivan ceo broj $n = n(x,y) > 1$ takav da je $(xy)^n = yx$ tada je R komutativan.

Recived by the editors September 20, 1987.