Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 21, 2 (1991), 91-109

Review of Research
Faculty of Science
Mathematics Series

THE STUDY OF LINEAR d-CONNECTIONS ON THE TOTAL SPACE OF A VECTOR BUNDLE $\xi = (E, \pi, M)(I)$

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Abstract

In this paper a study of the d-connection transformations on the total space E of a vector bundle $\xi = (E, \pi, M)$ is given. Is 2. the τ - and $\Omega(\rho)$ - systems of tensor equations on E are obtained, general solutions and some particular solutions are obtained, using the method given in [3]. Starting from these equations in 3. a general study of the connection transformations $\tau : D \to \overline{D}$ on E is developed. Furthermore, special formulas of the classical projective transformations, which preserve the class of linear d-connections on E are given. In 4. the transformations which preserve the torsion are studied.

The study will be continued in the second paper (II) in which the transformations of linear d-connections with the invariants of Schouten and Weyl type will be studied.

AMS Mathematics Subject Classification (1980): 53C60
Key words and phrases: τ-system, $\Omega(\rho)$-system, projective transformations, Weyl transformations, linear d-connections.

1. Introduction

In the present paper the notions and notations of R. Miron [1] and P. Stavre [3] are used. Let $\xi = (E, \pi, M)$ be a vector bundle, with an n-dimensional base manifold M, type fibre $F = R^n$ and a total space E of dimension $(n+m)$. Let N be a fixed non-linear connection whose existence is assured for example when M is paracompact, since in this case E is also paracompact. Then the tangent bundle T_uE in $u \in E$ has the unique decomposition $T_uE = H_uE \oplus V_uE$ and the tangent bundle to E is the Whitney sum $TE = HE \oplus VE$, where the horizontal bundle $HE \to E$ is considered as the horizontal distribution $H : u \in E \to H_uE$ and the vertical bundle $VE \to E$ as the vertical distribution $V : u \in E \to V_uE$. If N is fixed, then (H, V) is also fixed. We denote by h the horizontal projector and by v the vertical projector, determined by a fixed N. Let $\mathcal{X}(E)$ be the module of vector fields on E. Then $X \in \mathcal{X}(E)$ can be decomposed uniquely in the form: $X = hX + vY$, where hX is the horizontal part and vY is the vertical part, respectively. There also exists the dual decomposition: $T_u^*E = (H_uE)^* \oplus (V_uE)^*$, consequently for any 1-form field ρ on E we have the unique decomposition $\rho = h\rho + v\rho$, where $h\rho(X) = \rho(hX); v\rho(X) = \rho(vX)$.

Local coordinate transformations on E are given by:

$$(1.1) \qquad x^{i'} = x^{i'}(x^1, x^2, ..., x^n); \qquad \mathrm{rank}(\frac{\partial x^{i'}}{\partial x^i}) = n;$$

$$y^{a'} = M_a^{a'}(x) y^a; \qquad \mathrm{rank}(M_a^{a'}(x)) = m$$

$(i, i' = 1, ..., n : a, a' = 1, ...m)$, where $M_a^{a'}(x)$ is the matrix of the linear application $(g_{ba} o \varphi_a^{-1})(x)$, defined by the change of the vectorial maps and of the maps on M, which define the maps on E, g_{ba} being the structural functions. The local basis defined by $\delta/_{\delta x^i} = \varphi/_{\varphi x^i} - N_i^a \varphi/_{\varphi y^a}$, where $N_i^a(x, y)$ are the coefficients of the non-linear connection N, permits the existence of the dual basis $(dx^i, \delta y^a = dy^a + N_i^a dx^i)$. These local bases are the adapted bases for N and they play a prominent part in the study of linear d-connections, as is shown by R. Miron [1].

2. Systems of tensor equations on the total space E of the vector bundle $\xi = (E, \pi, M)$

Let h and v be the horizontal and the vertical projector determined by the non-linear connection N.

Definition 2.1. *System:*

(2.1) $$h\tau(X, vY) = 0$$

$\forall X, Y \in \mathcal{X}(E)$

(2.2) $$v\tau(X, hY) = 0$$

where τ is an unknown tensor field of the type (1.2) on E, is called a homogeneous τ-system of tensor equations on E.

Definition 2.2. *System*

(2.3) $$h\Omega(X, vY) = \rho(vY)hX \quad \forall X, Y \in \mathcal{X}(E)$$

(2.4) $$v\Omega(X, hY) = \rho(hY)vX$$

where Ω is an unknown tensor field of type (1.2) on E, and ρ is an 1-from on E, is called an $\Omega(\rho)$-system of tensor equations on E.

Such systems are studied in [3] as tensor-algebraic problems on E. In this section some special cases are studied, those which are necessary in the theory of d-connection transformations on the total space E of the vector bundle $\xi = (E, \pi, M)$.

Proposition 2.1. *A homogeneous $\tau-$ system cannot have solutions of the form:*

(2.5) $$\tau(X, Y) = \rho(X)Y = \rho(Y)X; \rho \neq o.$$

Proof. We admit that system (2.1), (2.2) has a solution of the form (2.5). In this case it follows from (2.1), (2.2) and (2.5) that $\rho(vY) = 0, \rho(hY) = 0, \forall Y \in \mathcal{X}(E)$. Consequently $v\rho = 0$ and $h\rho = 0$. Since we have the unique decomposition $\rho = h\rho + v\rho$, it follows that $\rho = 0$. This is a contradiction. Therefore a solution of a homogeneous τ-system cannot be of the form (2.5).

This result is very important in the theory of d-connections on E.

Proposition 2.2. *A special solution of a homogeneous $\tau-$ system (2.1), (2.2) is:*

$$\tau(X,Y) = \rho(X)Y : \forall X,Y \in \mathcal{X}(E). \tag{2.6}$$

Proof. We have: $\tau(X,vY) = \rho(X)v(Y)$ and analogously $\tau(X,hY) = \rho(X)hY$, $\forall X,Y \in \mathcal{X}(E)$. It follows that: $h\tau(X,vY) = 0$ and $v\tau(X,hY) = 0$. Consequently τ given by (2.6) is a solution of a τ-system (2.1), (2.2).

We shall see when a solution τ of a $\tau-$ system (2.1), (2.2) can be of the form:

$$\tau(X,Y) = \rho(X)Y + \rho(Y)X - \Omega(X,Y); \;\; \Omega \neq 0 \tag{2.7}$$

We have:

Theorem 2.1. *The necessary and sufficient condition that τ given by (2.7) be a solution of a homogeneous $\tau-$ system (2.1), (2.2), is that Ω be a solution of an $\Omega(\rho)-$ system (2.3),(2.4).*

Conversely: let Ω be the general solution of an $\Omega(\rho)-$ system (2.3),(2.4) and τ defined by (2.7) associated to this solution, we obtain:

$$\tau(X,vY) = \rho(X)vY + \rho(vY)X - \Omega(X,vY) \tag{2.8}$$

$$\tau(X,hY) = \rho(X)hY + \rho(hY)X - \Omega(X,vY). \tag{2.9}$$

From (2.3) and (2.8) it follows that: $h\tau(X,vY) = 0$. From (2.4) and (2.9) it follows that $v\tau(X,hY) = 0$. Q.e.d.

Consequently, if we look for solution of the form (2.7) for a homogeneous $\tau-$ system, then for any given ρ it follows that the arbitrarity degree of Ω is given by its generality as the solution of an $\Omega(\rho)$ system.

Theorem 2.2 is proved in [3].

Theorem 2.2. *The general solution Ω of an $\Omega(\rho)-$ system can be written in the form:*

$$\begin{aligned}\Omega(X,Y) &= \rho(vY)hX + \rho(hY)vX + h_1^D(hX,hY) + v_2^D(hX,vY) \\ &\quad + h_3^D(vX,hY) + v_4^D(vX,vY); \;\; \forall X,Y \in \mathcal{X}(E)\end{aligned} \tag{2.10}$$

where : $h_1^D(hX,hY); v_2^D(hX,vY); h_3^D(vX,hY) + v_4^D(vX,vY)$, are arbitrary d-tensor fields on E of the types: $\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}; \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}; \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}; \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}$ *respectively.*

We can also give other equivalent forms of (2.10), which are necessary in the applications.

Following easily is :

Proposition 2.3. *A particular solution of an $\Omega(\rho)$- system is given by Ω defined by:*

$$\Omega(X,Y) = -\rho(X)Y + \rho(Y)X; \ \ \forall X,Y \in \mathcal{X}(E) \tag{2.11}$$

By substituting (2.11) in (2.7) it follows that:

$$\tau(X,Y) = 2\rho(X)Y \tag{2.12}$$

is a solution of the form (2.7) of a τ-system. Noting $2\rho = \alpha$, we obtain:

Proposition 2.4. *Solution :*

$$\tau(X,Y) = \alpha(X)Y; \ \ \forall X,Y \in \mathcal{X}(E) \tag{2.13}$$

of a homogeneous τ-system is of the form (2.7), where $2\rho = \alpha$ and Ω is given by (2.11).

Using the decomposition $\rho = h\rho + v\rho$ and the relation (2.10), from (2.7) follows:

Theorem 2.3. *The most general solution τ of the form (2.7) for a τ-system with symmetric Ω, i.e. $\Omega(X,Y). = \Omega(Y,X); \forall X,Y \in \mathcal{X}(E)$, is given by:*

$$\begin{aligned} \tau(X,Y) = {} & \rho(hX)hY + \rho(hY)hX + \rho(vX)vY + \rho(vY)vX - \\ & - h_1^D(hX,hY) - v_4^D(vX,vY); \forall X,Y \in \mathcal{X}(E) \end{aligned} \tag{2.14}$$

where:

$$h_1^D(hX,hY) = h_1^D(hY,hX) \tag{2.15}$$

$$v_4^D(vX,vY) = v_4^D(vY,vX). \tag{2.16}$$

We can consider $\overset{D}{_1} = 0$ and $\overset{D}{_2} = 0$.

Especially important is the solution τ of the form (2.7) with the property $\Omega(X,Y) = -\Omega(Y,X)$, i.e. where Ω a $\mathcal{X}(E)$-valuted 2-form. There follows:

Theorem 2.4. *The τ solution of a τ-system of the form (2.7) with $\Omega(X,Y) = -\Omega(Y,X)$ has the form:*

$$\begin{aligned} &\tau(X,Y) = \rho(hx)hY + \rho(hY)hX + \rho(vX)vY + \rho(vY)vX + \\ (2.17)\quad &+2\rho(hX)vY + 2\rho(vX)hY) - h_1^D(hX,hY) - v_4^D(vX,vY) \\ &\forall X,Y \in \mathcal{X}(E) \end{aligned}$$

where:

$$(2.18)\qquad h_1^D(hX,hY) = -h_1^D(hY,hX) \quad \forall X,Y \in \mathcal{X}(E).$$

$$(2.19)\qquad v_4^D(vX,vY) = -v_4^D(vY,vX)$$

We can consider $\overset{D}{_1} = 0$ and $\overset{D}{_2} = 0$.

3. Connection transformations on the total space E

Using the differential manifold structure induced on E, the differential manifold structure of M and the vector bundle structure $\xi = (E,\pi,M)$, we can study in the usual way the geometry of the $(n+m)$ dimensional differentiable manifold E. But in the case when there exists a non-linear connection N (for example if M is paracompact, then E is also paracompact and there exists N), then we obtain some remarkable results in the geometry of the total space E of the vector bundle $\xi = (E,\pi,M)$.

In the case above specified, there exist distinguished connections on E, with special geometrical properties, i.e. those which preserve the horizontal distribution H and the vertical distribution V. These connections are called by R. Miron [1], linear d-connections on the total space E, and their theory is developed in [1].

If D is a linear d-connection on E, then equivalenly we have the conditions:

$$(3.1)\qquad vD_X hY = 0; hD_X vY = 0; \forall X,Y \in \mathcal{X}(E).$$

Let D be a fixed linear connection on E, then we call $E_{n+m} = (E,D)$ a linear connected space as in the general theory. If D is symmetric, it is known that: $T(X,Y) = T(Y,X)$, where T is the torsion of the connection D, i.e. $T(X,Y) = D_X Y - D_Y X - \{X,Y\}, \forall X,Y \in \mathcal{X}(E)$. Then the most general connection transformations $\tau : D \to \overline{D}$, which preserve the autoparallel curves of the space $E_{n+m} = (E,D)$, are the Weyl projective transformations:

$$\overline{D}_X Y = D_X Y + \rho(X)Y + \rho(Y)X; \forall X,Y \in \mathcal{X}(E). \tag{3.2}$$

If D is d-linear, a necessary condition for the existance is that the distribution H be integrable.

In the following we consider (3.2) in the more general case, when D not symmetric and we call the transformations $\tau; D \to \overline{D}$ given by (3.2), clasical Weyl projective transformations of linear connections.

Proposition 3.1. *The projective connection transformations (3.2) do not preserve the class of linear d-connection on E.*

Proof. We denote: $\tau(X,Y) = \rho(X)Y + \rho(Y)X$. Imposing the condition that D and $\overline{D}$ satisfy (3.1), we obtain the tensorial system (2.1.), (2.2). According to Proposition 2.1. this system does not admit solutions of the form (2.5). Consequently transformations (3.2) not preserve the class of linear d-connections.

There do not exist classical projective transformations (3.2) of symmetric linear d-connections, [3].

Corollary 3.1. *Let D and $\overline{D}$ be two symmetric linear connections on E, which preserve, by parallelism, the horizontal distribution H and the vertical distribution V on E. Then the spaces (E,D) and $(E,\overline{D})$ cannot have the same autoparallel curves. If these spaces have the same autoparallel curves, then $D = \overline{D}$.*

Definition 3.1. *The most general projective transformations, which preserve the autoparallel curves of the space, namely the transformations $\tau; D \to \overline{D}$ given by:*

$$\overline{D}_X Y = D_X Y + \rho(X)Y + \rho(Y)X - \Omega(X,Y); \quad \forall X,Y \in \mathcal{X}(E) \tag{3.3}$$

where $\Omega(X,Y) = -\Omega(Y,X)$, are called general projective transformations and are denoted by $\tau_{[\Omega]}(\rho)$.

Theorem 3.1. *The most general projective transformations, which preserve the class of linear d-connections on E are of the form:*

$$(3.4)\quad \begin{aligned}\overline{D}_X Y &= D_X Y + \rho(hX)hY + \rho(hY)hX + \rho(vX)vY + \rho(vY)vX + \\ &+2\rho(hX)vY + 2\rho(vX)hY - h_1^D(hX,hY) - v_4^D(vX,vY)\end{aligned}$$

where the tensorial d-fields $h_1^D(hX,hY), v_1^D(vX,vY)$ satisfy the relations:

$$(3.5)\quad h_1^D(hX,hY) = -h_1^D(hY,hX)$$

$$\forall X,Y \in \mathcal{X}(E).$$

$$(3.6)\quad v_4^D(vX,vY) = -v_4^D(vY,vX)$$

Proof. If we denote:

$$(3.7)\quad \tau(X,Y) = \rho(X)Y + \rho(Y)X - \Omega(X,Y)$$

in (3.3) and put the condition that D and $\overline{D}$ be d-linear, we obtain system (2.1), (2.2), which admits necessarely a solution of the form (2.7) with $\Omega(X,Y) = -\Omega(Y,X)$ or equivalently system (2.3),(2.4) admits the solution $\Omega \neq 0$ with the property $\Omega(X,Y) = -\Omega(Y,X)$. According to Theorem 3.3, it follows that τ is of the form (2.17), (2.18) and (2.19). Consequently we obtain (3.4).

We shall denote transformations (3.3) by $\tau_{[\Omega]}(\rho)$. In applications it is more confortable to write these transformations in another form and so there follows:

Theorem 3.2. *The transformations $\tau_{[\Omega]}(\rho)$ which preserve the class of linear d-connections on E are of the form:*

$$(3.8)\quad \overline{D}_{hX}hY = D_{hX}hY + \rho(hX)hY + \rho(hY)hX - h_1^D(hX,hY)$$

$$(3.9)\quad \overline{D}_{hX}vY = D_{hX}vY + 2\rho(hX)vY$$

$$\forall X,Y \in \mathcal{X}(E)$$

$$(3.10)\quad \overline{D}_{vX}hY = D_{vX}hY + 2\rho(vX)hY$$

$$(3.11)\quad \overline{D}_{vX}vY = D_{vX}vY + \rho(vX)vY + \rho(vY)vX - v_4^D(vX,vY)$$

where $\overset{D}{1}$ and $\overset{D}{4}$ satisfy (3.5) and (3.6). If we consider:

$$h_1^D(hX,hY) = \rho(hY)hX - \rho(hX)hY \tag{3.12}$$

$$v_4^D(vX,vY) = \rho(vY)vX - \rho(vX)vY \tag{3.13}$$

there follows the Weyl transformations:

$$\overline{D}_{hX}hY = D_{hX}hY + \alpha(hX)hY \tag{3.14}$$

$$\begin{gathered}\overline{D}_{hX}vY = D_{hX}vY + \alpha(hX)vY \\ \forall X,Y \in \mathcal{X}(E)\end{gathered} \tag{3.15}$$

$$\overline{D}_{vX}hY = D_{vX}hY + \alpha(vX)hY \tag{3.16}$$

$$\overline{D}_{vX}vY = D_{vX}vY + \alpha(vX)vY \tag{3.17}$$

where we denote $\alpha = 2\rho$. Or, considering $\Omega(X,Y) = \rho(Y)X - \rho(X)Y$, we have $\Omega(X,Y) = -\Omega(Y,X)$ and Ω is a solution of an $\Omega(\rho)$-system. Consequently we obtain the Weyl transformations:

$$\overline{D}_XY = D_XY + \alpha(X)Y \quad \forall X,Y \in \mathcal{X}(E) \tag{3.18}$$

which preserve the class of linear d-connections. We have:

Theorem 3.3. *The Weyl transformations are a special case of transformations (3.4).*

Obviously if we put $\rho = 0$ in (2.3) and (2.4) there follows a homogeneous τ-system, with the solutions of the form:

$$\begin{gathered}\tau(X,Y) = h_1^D(hX,hY) + v_2^D(hX,vY) + \\ + h_3^D(vX,hY) + v_4^D(vX,VY)\end{gathered} \tag{3.19}$$

which is the general solution of a homogeneous τ- system. There follows:

Theorem 3.4. *The most general connection transformations $\tau : D \to \overline{D}$, which preserve the class of linear d-connection on E are of the form:*

$$\begin{gathered}\overline{D}_XY = D_XY + h_1^A(hX,hY) + v_2^A(hX,vY) + \\ + h_3^A(vX,hY) + v_4^A(vX,vY) \\ \forall X,Y \in \mathcal{X}(E)\end{gathered} \tag{3.20}$$

where $h_1^A(hX,hY), v_2^A(hX,vY), h_3^A(vX,hY), v_4^A(vX,vY)$ are arbitrary d-tensor fields on E.

Theorem 3.5. *Let N be a fixed non-linear connection on $\xi = (E, \pi, M)$ and D a linear d-connection relative to N. Then the set of all linear d-connections relative to N on E, is given by (3.20).*

Their group is a group isomorphic with the aditive group of the tensor fields of the form (3.19), relative to the composition of transformations.

We denote by $\mathcal{A}T(E) = \bigoplus DT^{rs}_{pq}$ the algebras of the d-tensor fields on E, where DT^{rs}_{pq} is the module the d-tensor fields of the type $\begin{pmatrix} r & s \\ p & q \end{pmatrix}$ and $\mathcal{A}T(E)$ is endowed with the tensorial product $T_1 \otimes T_2$ of the d-tensor fields. The fact that the $\mathcal{A}T(E)$ algebra can be locally identified with the tensor algebra $\mathcal{A} = \bigoplus \tau^{rs}_{pq}$ of the vector bundle $HE \bigoplus VE \to E$ [1], leads to an elegant theory of the above introduced notions by the use of a local adapted basis $(\delta/\delta x^i = \partial/\partial x^i - N^a_i \partial/\partial y^a;\ \partial/\partial y^b;\ \ i = 1, ..., n,\ \ a, b = 1...m)$, where $N^a_i(x, y)$ are the coefficients of the non-linear connection N.

If we denote $(X_i = \delta/\delta x^i, ; X_{n+\alpha} = \partial/\partial y^a) = \{X_\alpha\}, \alpha = 1, ..., n, n + 1, ..., n + m)$ and :

$$(3.21) \qquad D_{X_\beta} X_\alpha = \Gamma^\sigma_{\alpha\beta} X_\delta : (\alpha, \beta, \sigma = 1, ...n, n + 1, ..., n + m)$$

then, if D is a linear d-connection, we have :

$$(3.22) \qquad \begin{array}{c} \Gamma^{n+c}_{ij}(x, y) = 0; \Gamma^k_{n+a,j}(x, y) = 0; \\ \Gamma^{n+c}_{i\ \ n+b} = 0;\ \ \Gamma^k_{n+a n+b} = 0 \end{array}$$

$(i, j, k = 1...n;\ a, b, c = 1...m)$. Consequently, remain four coefficients, Γ, generally different from zero.

From the unique decomposition $X = hX + vX,\ \ X \in \mathcal{X}(E)$ we have : $hX = X^i \delta/\delta x^i,\ \ vX = X^{n+a} \partial/\partial y^a$, which are d-tensor fields of the type $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ respectively, on E. If we denote $\tau = h\rho, ; \sigma = v\rho$, which are tensorial d- fields of the type $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ respectively from the unique decomposition $\rho = h\rho + v\rho$, locally we can write: $\tau = \tau_i dx^i,\ \ \sigma = \sigma_{n+a} \delta y^a (i = 1, 2, ..., n,\ \ a = 1, 2, ..., m)$ relative to the basis $(dx^i, \delta y^a = dy^a + N^a_i dx^i)$ which is the dual basis of the adapted basis.

We denote:

$$(3.23) \qquad h^A_1(X_\beta, X_\alpha) = A^i_{\alpha\beta} \delta/\delta x^i;\ \ v^A_4(X_\beta, X_\alpha) = B^{n+c}_{\alpha\beta} \partial/\partial y^c.$$

There follows:

$$h_1^A(X_k, X_j) = A^i_{jk}\delta/\delta x^i;$$
$$v_4^A(X_{n+b}, X_{n+a}) = B^{n+c}_{n+a\,n+b}\partial/\partial y^c. \tag{3.24}$$

According to the above indicated identification we obtain the d-tensor fields $A, B \in \mathcal{A}$ of the form :

$$A = A^i_{jk}\delta/\delta x^i \bigotimes dx^j \bigotimes dx^k \tag{3.25}$$

$$B = B^{n+c}_{n+a\,n+b}\partial/\partial y^c \bigotimes \delta y^a \bigotimes \delta y^b. \tag{3.26}$$

From Theorem 3.2, with these notations there follows :

Theorem 3.6. *The most general $\tau_{[\Omega]}(\acute{\rho})$ transformations, which preserve the class of linear d-connections are of the form:*

$$\overline{\Gamma}^i_{jk} = \Gamma^i_{jk} + \tau_k\delta^i_j + \tau_j\delta^i_k - A^i_{jk}; \overline{\Gamma}^{n+c}_{jk} = \Gamma^{n+c}_{jk} \tag{3.27}$$

$$\overline{\Gamma}^{n+c}_{n+a\ k} = \Gamma^{n+c}_{n+a\ k} + 2\tau_k\delta^{n+c}_{n+a};\ \overline{\Gamma}^i_{n+a\ k} = \Gamma^i_{n+a\ k} \tag{3.28}$$

$$\overline{\Gamma}^i_{j\,n+b} = \Gamma^i_{j\,n+b} + 2\sigma_{n+b}\delta^i_j;\ \overline{\Gamma}_j^{n+c}{}_{n+b} = \Gamma_j^{n+c}{}_{n+b} \tag{3.29}$$

$$\overline{\Gamma}^{n+c}_{n+a\,n+b} = \Gamma^{n+c}_{n+a\,n+b} + \sigma_{n+b}\delta^{n+c}_{n+a} +$$
$$+\sigma_{n+a}\delta^{n+c}_{n+b} - B^{n+c}_{n+a\,n+b};\ \overline{\Gamma}^i_{n+a\,n+b} = \Gamma^i_{n+a\,n+b} \tag{3.30}$$

where $\tau_k, \sigma_k, A^i_{jk}$ and B^i_{jk} are d-tensor fields of the type $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$ *and* $\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}$ *respectively, which satisfy the relations:*

$$A^i_{jk} = -A^i_{kj};\quad B^{n+c}_{n+a\,n+b} = -B^{n+c}_{n+b\,n+a} \tag{3.31}$$

Theorem 3.7. *Let D be a fixed linear d-connection on E with the local coefficients Γ in the adapted basis, and $\bar{D}$ an arbitrary d-connection with the coefficients $\bar{\Gamma}$ relative to the same adapted basis. Then $\bar{D}$ is obtained from D by a general projective transformation $\tau_{[\Omega]}(\rho)$, if and only if between Γ and $\bar{\Gamma}$ the following relations hold:*

$$\overline{\Gamma}^i_{jk} = \Gamma^i_{jk} + \tau_k\delta^i_j + \tau_j\delta^i_k - A^i_{jk} \tag{3.32}$$

$$\overline{\Gamma}^{n+c}_{n+a\ k} = \Gamma^{n+c}_{n+a\ k} + 2\tau_k\delta^{n+c}_{n+a} \tag{3.33}$$

$$\overline{\Gamma}^{i}_{jn+b} = \Gamma^{i}_{jn+b} + 2\sigma_{n+b}\delta^{i}_{j} \tag{3.34}$$

$$\overline{\Gamma}^{n+c}_{n+an+b} = \Gamma^{n+c}_{n+an+b} + \sigma_{n+b}\delta^{n+c}_{n+a} + \sigma_{n+a}\delta^{n+c}_{n+b} - B^{n+c}_{n+an+b} \tag{3.35}$$

where A and B satisfies (3.31) .

We can consider $A = 0,\ B = 0$.

Very important are also the transformations (3.3) with the property $\Omega(X,Y) = \Omega(Y,X), \forall X,Y \in \mathcal{X}(E)$, which will be called $\tau_{\{\Omega\}}(\rho)-$ transformations. We have:

Theorem 3.8. *The most general linear connection transformation $\tau_{\{\Omega\}}(\rho)$ on E, which concerves the class of linear d−connections, is of the form:*

$$\begin{aligned}\overline{D_X}Y = D_XY + \rho(hX)hY + \rho(hY)hX + \rho(vX)vY + \rho(vY)vX - \\ h^D_1(hX,hY) - v^D_4(vX,vY) \quad \forall X,Y \in \mathcal{X}(E)\end{aligned} \tag{3.36}$$

where

$$h^D_1(hX,hY) = h^D_1(hY,hX); \quad \forall X,Y \in \mathcal{X}(E) \tag{3.37}$$

$$v^D_4(vX,vY) = v^D_4(vY,vX). \tag{3.38}$$

Proof. If we denote $\tau(X,Y) = \rho(X)Y + \rho(Y)X - \Omega(X,Y)$, since $\Omega(X,Y) = \Omega(Y,X)$ we have equivalently $\tau(X,Y) = \tau(Y,X)$. From these and form Theorem 3.2 there follows (3.36), (3.37) and (3.39). Equivalently we have:

Theorem 3.9. *The most general $\tau_{\{\Omega\}}(\rho)$ transformation, which preserves the class of linear d− connections on E, is of the form:*

$$\overline{D}_{hX}hY = D_{hX}hY + \rho(hX)hY + \rho(hY)hX - h^D_1(hX,hY) \tag{3.39}$$

$$\overline{D}_{hX}vY = D_{hX}vY; \qquad \forall X,Y \in \mathcal{X}(E) \tag{3.40}$$

$$\overline{D}_{vX}hY = D_{vX}hY \tag{3.41}$$

$$\overline{D}_{vX}vY = D_{vX}vY + \rho(vX)vY + \rho(vY)vX - v^D_4(vX,vY) \tag{3.42}$$

with the conditions (3.37) and (3.38) .

Definition 3.2. *A $\tau_{\{\Omega\}}(\rho)$ transformation with $\overset{D}{1} = 0$ and $\overset{D}{2} = 0$ is called a projective $(hh - vv)-$ transformation.*

With the local notations we have:

Theorem 3.10. *The most general $\tau_{\{\Omega\}}-$ transformations, which preserve the class of $d-$connections on E is given locally by:*

$$\overline{\Gamma}^{i}_{jk} = \Gamma^{i}_{jk} + \tau_k\delta^i_j + \tau_j\delta^i_k - A^i_{jk}; \; \overline{\Gamma}^{n+c}_{jk} = \Gamma^{n+c}_{jk} \tag{3.43}$$

$$\overline{\Gamma}^{n+c}_{n+a\;k} = \Gamma^{n+c}_{n+a\;k}; \quad \overline{\Gamma}^{i}_{n+a\;k} = \Gamma^{i}_{n+a\;k} \tag{3.44}$$

$$\overline{\Gamma}^{i}_{jn+b} = \Gamma^{i}_{jn+b}; \quad \overline{\Gamma}^{n+c}_{j\;\;n+b} = \Gamma^{n+c}_{j\;\;n+b} \tag{3.45}$$

$$\overline{\Gamma}^{n+c}_{n+an+b} = \Gamma^{n+c}_{n+an+b} + \sigma_{n+b}\delta^{n+c}_{n+a} + \sigma_{n+a}\delta^{n+c}_{n+b} - B^{n+c}_{n+an+b}; \tag{3.46}$$

$$\overline{\Gamma}^{i}_{n+an+b} = \Gamma^{i}_{n+an+b}$$

where the arbitrary $d-$tensor fields A and B of the type $\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$ *and* $\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}$ *respectively are symmetric:*

$$A^i_{jk} = A^i_{kj}; \quad B^{n+c}_{n+bn+a} = B^{n+c}_{n+bn+a}. \tag{3.47}$$

We can consider $A = 0$ and $B = 0$ and we obtain the local form of the $(hh - vv)-$projective transformations.

4. Linear $d-$connection transformations with invariant $\overline{T} = T$

In this paragraph transformations which preserve the class of linear $d-$connections and have some invariants associated to the torsions of these connections are determined. Firstly we consider the transformations $\tau : D \to \overline{D}$,

$$\overline{D}_X Y = D_X Y + \tau(X,Y); \quad \forall X,Y \in \mathcal{X}(E) \tag{4.1}$$

where τ is of the form

$$\tau(X,Y) = \rho(X)Y + \rho(Y)X - \Omega(X,Y).$$

Let T be the torsion of D and $\overline{T}$ the torsion of $\bar{D}$. There follows:

Theorem 4.1. *The most general transformations $\tau_\Omega(\rho)$ which preserve the torsion and the class of linear d−connections on E are the $\tau_{\{\Omega\}}(\rho)$− transformations.*

Proof. We have $T = \overline{T}$ and equivalently $\tau(X,Y) = \tau(Y,X)$ and $\Omega(X,Y) = \Omega(Y,X)$. It follows from Theorem 3.9 that these transformations are $\tau_{\{\Omega\}}(\rho)$− transformations.

Let D be a fixed linear d− connection on E and T its torsion. Since $T(X,Y) = -T(Y,X)$ and D is linear, we have (3.1), and it follows that T is characterized by the following d−tensor fields [1]:

(4.2) $$hT(hX,hY) = D_{hX}hY - D_{hY}hX - h[hX,hY]$$

(4.3) $$vT(hX,hY) = \Omega(X,Y)$$

(4.4) $$hT(hX,vY) = -D_{vY}hX - h[hX,vY] \ \forall X,Y \in \mathcal{X}(E)$$

(4.5) $$vT(hX,vY) = D_{hX}vY - v[hX,vY]$$

(4.6) $$hT(vX,vY) = 0$$

(4.7) $$vT(vX,vY) = D_{vX}vY - D_{vY}vX - v[vX,vY]$$

where Ω is the curvature 2-from of the non-linear connection. Since the vertical distribution v is integrable, we have $h[vX,vY] = 0$ and $v[vX,vY] = [vX,vY]$.

Proposition 4.1. *Let D be a fixed linear d−connection on E. A necessary and sufficient condition that any linear d−connection on E with the properties:*

(4.8) $$h\overline{T}(hX,vY) = hT(hX,vY); \quad v\overline{T}(hX,vY) = vT(hX,vY); \ \forall X,Y \in \mathcal{X}(E)$$

has the same torsion with $D(\overline{T} = T)$ and be obtained from D by a $\tau_\Omega(\rho)$− transformation, is that the relations:

(4.9) $$h\overline{D}(hX,vY) = hD(hX,hY) + \rho(hX)hY + \rho(hY)hX - hA(hX,hY)$$

(4.10) $$v\overline{D}(vX,vY) = vD(vX,vY) + \rho(vX)vY + \rho(vY)vX - vB(vX,vY) \quad \forall X,Y \in \mathcal{X}(E)$$

hold, where the d−tensor fields on E satisfy the relations:

(4.11) $$hA(hX,hY) = hA(hY,hX); \quad vB(vX,vY) = vB(vY,vX).$$

Proof. Since D and $\bar{D}$ are linear $d-$connections, from (4.9) and (4.10) it follows (3.39) and (3.42). From (4.8), (4.4) and (4.5) applied to D and $\bar{D}$ it follows (3.40) and (3.41). Consequently $\bar{D}$ is obtained from D by a $\tau_{\{\Omega\}}(\rho)-$ transformation, which is a $\tau_{\Omega}(\rho)-$ transformation which preserves $\bar{T} = T$.

Otherwise from (4.9) and (4.10) it follows that :

$$\begin{aligned} h\bar{T}(hX,hY) &= hT(hX,hY); \\ v\bar{T}(vX,vY) &= vT(vX,vY); \quad \forall X,Y \in \mathcal{X}(E) \end{aligned} \tag{4.12}$$

Since D and $\bar{D}$ are linear $d-$connections from (4.8), (4.12) and (4.3)-(4.7) it follows that $\bar{T} = T$ and we have (3.39)-(3.42).

We can observe, that if D and $\bar{D}$ are linear $d-$connections on E, with the properties (4.8), (4.12), then $\bar{T} = T$. From (4.8) it follows (3.40), (3.41), but not necessary also (3.30) and (3.33). In these conditions from (3.20) it follows

$$\overset{A}{v_2}(hX,vY) = 0, \quad \overset{A}{h_3}(vX,hY) = 0 \tag{4.13}$$

There follows :

Theorem 4.2. *The most general transformations $\tau : D \to \bar{D}$, which preserve the class of linear $d-$connections on E and the torsion $\bar{T} = T$ are given by:*

$$\bar{D}_X Y = D_X Y + \overset{A}{h_1}(hX,hY) + \overset{A}{v_4}(vX,vY); \quad \forall X,Y \in \mathcal{X}(E) \tag{4.14}$$

where the $d-$tensor fields $\overset{A}{_1}$ and $\overset{A}{_2}$ satisfy (4.13), but are otherwise arbitrary.

Relative to a local basis adaptated to N, Theorem 4.1 can be written in the form:

Proposition 4.2. *Let D be a fixed linear $d-$connection on E with the coefficients Γ. A necessary and sufficient condition that any other linear $d-$connection $\bar{D}$ on E with the coefficients $\bar{\Gamma}$ and the property (4.8) has the same torsion $\bar{T} = T$ with D and be obtained from D by a $\tau_{\Omega}(\rho)-$ transformation, is that we have:*

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \tau_k \delta^i_j + \tau_j \delta^i_k - A^i_{jk} \quad (i,j,k = 1,...,n) \tag{4.15}$$

$$\begin{aligned} \Gamma^{n+c}_{n+a\,n+b} &= \Gamma^{n+c}_{n+a\,n+b} + \sigma_{n+b}\delta^{n+c}_{n+a} + \\ &\sigma_{n+a}\delta^{n+c}_{n+b} - B^{n+c}_{n+a\;\;n+b} (a,b,c = 1...m) \end{aligned} \tag{4.16}$$

where A and B satisfy (3.47) and $\tau = h\rho$, $\sigma = v\rho$. *Particularly we can have also* $A = 0$, $B = 0$.

A special case is that of the transformations $\tau : D \to \bar{D}$ which preserve the linear $d-$connections on E and are symmetric ($T = 0$, $\bar{T} = 0$.) There exists a symmetric linear $d-$connection on E only if the H distribution is integrable, i.e. the curvature 2-form of the non- linear connection N vanishes ($\Omega = 0$). But H is integrable when $vT(hX, hY) = 0$ for any linear $d-$connection D on E. There follows:

Theorem 4.3. *Let* $\xi = (E, \pi, M)$ *be a vector bundle and N a fixed non-linear connection with integrable H distribution. Then the most general* $\tau_\Omega(\rho)-$ *transformations, which preserve the class of linear d−connections and the torsion T are the* $\tau_{\{\Omega\}}(\rho)-$ *transformations.*

If D is $d-$linear symmetric ($T = 0$), then $\bar{D}$ is also $d-$linear symmetric ($\bar{T} = 0$).

The above given Theorems and Properties can be written analogously also for the conservation of symmetric linear $d-$connections, but the condition that H is integrable will be increased.

For a symmetric linear $d-$connection on E it follows from (4.4) and (4.5):

(4.17) $D_{hX}vY = v[hX, vY]; \quad D_{vX}hY = h[hY, vX]; \quad \forall X, Y \in \mathcal{X}(E)$

Consequently we have:

Proposition 4.3. *Let* $\xi = (E, \pi, M)$ *be a vector bundle and N a fixed non-linear connection with integrable H distribution. Then for every two symmetric linear d-connection on E we have relations (3.40) and (3.41).*

There follows

Theorem 4.4. *Let* $\xi = (E, \pi, M)$ *be a vector bundle and N a fixed non-linear connection with integrable H distribution. The most general* $\tau_\Omega(\rho)-$ *transformations of symmetric linear d−connections on E are of the form (4.9), (4.10), (4.11) or locally of the form (4.15), (4.16), where A and B satisfy the condition (3.47).*

But these transformations are $\tau_{\{\Omega\}}(\rho)-$ transformations. Consequently a $\tau_{\{\Omega\}}(\rho)-$ transformation of symmetric linear $d-$connections on E is characterized by the transformations (4.9),(4.10) and (4.11) or locally by (4.15),(4.16) and (3.47).

Particularly we can consider $A = 0$ and $B = 0$ and we obtain the projective $(hh - vv)-$transformations of symmetric linear $d-$connections:

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \tau_j\delta^i_k + \tau_k\delta^i_j. \tag{4.18}$$

$$\bar{\Gamma}^{n+c}_{n+a\,n+b} = \Gamma^{n+c}_{n+a\,n+b} + \sigma_{n+a}\delta^{n+c}_{n+b} + \sigma_{n+b}\delta^{n+c}_{n+a} \tag{4.19}$$

where $\tau = h\rho$ and $\tau = v\rho$ and D is a fixed symmetric linear $d-$connection on E.

Consequently the projective $(hh - vv)-$transformations are $\tau_{\{\Omega\}}(\rho)-$ transformations with $\underset{1}{D} = 0$ and $\underset{2}{D} = 0$, or locally of the form (3.43),(3.44),(3.45) and (3.46) in which we consider $A = 0$, $B = 0$ and $\tau = h\rho$, $\sigma = v\rho$. It follows that there are $\tau_\Omega(\rho)-$ transformations $(\bar{D}_XY = D_XY + \tau(X,Y))$, where τ is of the form (2.7) with Ω of the form:

$$\Omega(X,Y) = \rho(vX)hY + \rho(hX)vY + \rho(hY)vX + \rho(vY)hX;\ \ \forall X,Y \in \mathcal{X}(E) \tag{4.20}$$

where we denote $\tau = h\rho$, $\sigma = v\rho$, which are $d-$tensor fields of the type $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ respectively on E. These can be obtained also as a special case from (4.14).

If in (4.14) we choose:

$$h_1^A(hX,hY) = \tau(X)hY + \tau(Y)hX - hA(hX,hY) \tag{4.21}$$

$$v_4^A(vX,vY) = \sigma(X)vY + \sigma(Y)vX - vB(vX,vY) \tag{4.22}$$

$$\forall X,Y \in \mathcal{X}(E)$$

without requiring that the $d-$tensor fields τ of the type $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and σ of the type $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, be of the form $\tau = h\rho$ and $\sigma = v\rho$, but completly arbitrary, we obtain the transformations:

$$\begin{aligned} \bar{D}_XY = D_XY + \tau(X)hy + \tau(Y)hX + \sigma(X)vY + \sigma(Y)vX - \\ - hA(hX,hY) - vB(vX,vY)\ \ \forall X,Y \in \mathcal{X}(E). \end{aligned} \tag{4.23}$$

Definition 4.1. *Transformations (4.23) with $A = 0$ and $B = 0$ will be called $(hh - vv)-$transformations and will be denoted by $\tau_{\{hh-vv\}}$.*

Following next is:

Proposition 4.4. *Transformations (4.23) preserve the class of linear $d-$connections and its torsions. Particularly the $(hh-vv)-$ transformations $\tau_{\{hh-vv\}}$ preserve the class of linear $d-$connections and its torsion. They are of the form:*

$$\bar{D}_X Y = D_X Y + \tau(X)hY + \tau(Y)hX + \sigma(X)vY + \sigma(Y)vX; \quad \forall X, Y \in \mathcal{X}(E) \tag{4.24}$$

where τ and σ are arbitrary $d-$tensor fields on E of the type $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ respectively.

Proposition 4.5. *Let $\xi = (E, \pi, M)$ be a vector bundle and N a fixed non-linear connection, with integrable horizontal distribution H. For every two symmetric linear $d-$connections D and $\bar{D}$ on E, $\bar{D}$ is obtained locally from D by a $\tau_{\{hh-vv\}}-$ transformation if and only if between the local coefficients Γ and $\bar{\Gamma}$ there exist the relations (4.18),(4.19) where $\tau_j = \tau(\delta/\delta x^j)$; $\tau_{n+a} = \sigma(\partial/\partial y^a)$.*

Corollary 4.1. *If in (4.24) we take $\tau = h\rho$ and $\sigma = v\rho$ we obtain the projective $(hh - vv)-$ transformations.*

In a paper which is to follow we will study the transformations which preserve the linear $d-$connections on E and have invariants of Schouten type and some invariants associated to the curvature of the connection.

References

[1] Miron, R., Anastasiei, M.: Fibrate vectoriale. Spatii Lagrange. Aplicatii in Teoria relativitatii, Editura Acad. Romania, Bucuresti, 1987.

[2] Schouten, J.A.: Ricci-Calculus. Springer Verlag Berlin 1958.

[3] Stavre, P., Virlan, I.: Principal system of tensor equations. Analele Univ. Craiova, XVI (1988), 38-44.

REZIME

STUDIJA LINEARDNE d-KONEKSIJE NA TOTALNOM PROSTORU VEKTORSKOG SNOPA $\xi = (E, \pi, M)(I)$

U ovom radu se ispituje transformacija d-koneksije na totalnom prostoru E vektorskog snopa $\xi = (E, \pi, M)$. U 2. su dobiveni τ i $\Omega(\rho)$ sistemi tenzorskih jednačina na E kao i njihova opšta i partikularna rešenja (koristeći metode iz [3]). Polazeći od ovih jednačina u 3. je dato kompletno ispitivanje transformacija koneksija $\tau : D \to \vec{D}$ definisano nad E. Date su specijalne formule klasične projektivne transformacije koje ne menjaju klasu linearne d-koneksije nad E. U 4. su date transformacije koje ne menjaju torziju.

Ispitivanje će se nastaviti u radu (II) u kojem će se ispitati one transformacije linearne d-koneksije koje imaju invarijante tipa Schoutena i Weyla.

Received by the editors May 15, 1991