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COMPARATIVE CHARACTERIZATION OF GENERALIZED LAGRANGE AND GENERALIZED HAMILTON SPACES II

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Abstract

This paper is the continuation of [6]. The notations, and definitions given there will be used. In [6] the generalized Lagrange and Hamilton spaces were introduced and the torsion and the connection coefficients for recurrent spaces were determined. Here the curvature theory and Ricci and Bianchi equations for these spacese will be given.

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1. Ricci identities in generalized Lagrange spaces

In the tangent space $T(E)$ of the generalized Lagrange space $L^{n+m} = (E, M, N, G, \nabla, T, \lambda)$ there are given vector fields

$$X = X^i\delta_i + X^a\partial_a, \quad Y = Y^j\delta_j + Y^b\partial_b, \quad Z = Z^k\delta_k + Z^c\partial_c$$

$$i, j, k, h, l = \overline{1, n}, \qquad a, b, c, d, e = \overline{a, m}.$$

The curvature tensor R is defined as usually by

$$(1.1)\ R = R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z = R^k\delta_k + R^c\partial_c.$$

Using the linearity of ∇ and Definition 4.1. from [6] we obtain

Theorem 1.1. *The curvature tensor in the generalized Lagrange space has the form:*

$$(1.2)\qquad \begin{aligned} R^x = \; & 2^{-1} \; (R_{e\ ji}^{\ x} Z^e + R_{d\ ji}^{\ x} Z^d)(X^i Y^j - Y^i X^j) + \\ & (P_{e\ ja}^{\ x} Z^e + P_{d\ ja}^{\ x} Z^d)(X^a Y^j - Y^a X^j) + \\ & 2^{-1} \; (S_{e\ ba}^{\ x} Z^e + S_{d\ ba}^{\ x} Z^d)(X^a Y^b - Y^a X^b) \\ & x \in \{k, c\}, \end{aligned}$$

where

$$(1.3)\qquad \begin{aligned} &(a)\quad 2^{-1} R_{y\ ji}^{\ x} &&= \delta_{[i} F_{|y|\ j]}^{\ x} + F_{y\ [j}^{\ e} F_{|e|\ i]}^{\ x} \\ & &&\quad + F_{y\ [j}^{\ h} F_{|h|\ i]}^{\ x} + 2^{-1} C_{yb}^{x} K_{ji}^{b} \\ &(b)\quad 2^{-1} K_{ji}^{b} &&= \partial_{[i} N_{j]}^{b} + N_{[j}^{\ a} \partial_a N_{i]}^{b} \quad K_{aj}^{\ b} = \partial_a N_j^b - F_{a\ j}^{\ b} \\ &(c)\quad P_{y\ ja}^{\ x} &&= \partial_a F_{y\ j}^{\ x} - C_{y\ a|j}^{\ x} + K_{aj}^{\ b} C_{y\ b}^{\ x} \\ &(d)\quad 2^{-1} S_{y\ ba}^{\ x} &&= \partial_{[b} C_{|y|\ a]}^{\ x} + C_{y\ [a}^{\ h} C_{|h|\ b]}^{\ x} + C_{y\ [a}^{\ d} C_{|d|\ b]}^{\ x} \\ &(e)\quad C_{y\ a|j}^{\ x} &&= \delta_j C_{y\ a}^{\ x} + C_{y\ a}^{\ d} F_{d\ j}^{\ x} + C_{y\ a}^{\ h} F_{h\ j}^{\ x} \\ & &&\quad - C_{d\ a}^{\ x} F_{y\ j}^{\ d} - C_{h\ a}^{\ x} F_{y\ j}^{\ h} - C_{y\ d}^{\ x} F_{a\ j}^{\ d} \end{aligned}$$

and $x \in \{c, k\}$ $y \in \{d, l\}$.

On the other hand starting from

$$(1.4)\qquad \nabla_X Y = (Y^j{}_{|i} X^i + Y^j|_a X^a)\delta_j + (Y^b{}_{|i} X^i + Y^b|_a X^a)\partial_b,$$

where

$$(1.5)\qquad \begin{aligned} &(a)\quad Y^x{}_{|i} &&= \delta_i Y^x + F_{h\ i}^{\ x} Y^h + F_{d\ i}^{\ x} Y^d \\ &(b)\quad Y^x|_a &&= \partial_a Y^x + C_{h\ a}^{\ x} Y^h + C_{d\ a}^{\ x} Y^d \quad x = j \text{ or } \quad x = b, \end{aligned}$$

we get

Theorem 1.2. *The curvature tensor in the generalized Lagrange space has the form:*

$$(1.6)\qquad \begin{aligned} R^x \; = \; & A^x{}_{ji}(X^i Y^j - Y^i X^j) + A^x{}_{ja}(X^a Y^j - Y^a X^j) + \\ & A^x{}_{ba}(X^a Y^b - Y^a X^b), \end{aligned}$$

where

$$(1.7)\qquad \begin{aligned} A^x{}_{ji} &= Z^x_{\|[j|i]} + F_{[j}{}^h{}_{i]} Z^x{}_{|h} + (F_{[j}{}^d{}_{i]} + 2^{-1} K_j{}^d{}_i) Z^x|_d \\ A^x{}_{ja} &= Z^x{}_{|j}|_a - Z^x|_{a|j} + (C_j{}^h{}_a - F_a{}^h{}_j) Z^x{}_{|h} + \\ &\quad (C_j{}^d{}_a + K_a{}^d{}_j) Z^x|_d \\ A^x{}_{ba} &= Z^x_{[}|_b|_{a]} + C_{[b}{}^j{}_{a]} Z^x{}_{|j} + C_{[b}{}^d{}_{a]} Z^x|_d \\ &\quad x \in \{c, k\} \end{aligned}$$

Comparing (1.2) with (1.6) and (1.7) we get:

Theorem 1.3. *The Ricci equations for the generalized Lagrange spaces are:*

$$(1.8)\qquad \begin{aligned} &(a)\quad Z^x_{\|[j|i]} + F_{[j}{}^h{}_{i]} Z^x{}_{|h} + (F_{[j}{}^d{}_{i]} - 2^{-1} K_i{}^d{}_j) Z^x|_d = \\ &\qquad 2^{-1}(R_l{}^x{}_{ji} Z^l + R_d{}^x{}_{ji} Z^d) \\ &(b)\quad Z^x{}_{|j}|_a - Z^x|_{a|j} + (C_j{}^h{}_a - F_a{}^h{}_j) Z^x{}_{|h} + (C_j{}^d{}_a + K_a{}^d{}_j) Z^x|_d = \\ &\qquad P_l{}^x{}_{ja} Z^l + P_d{}^x{}_{ja} Z^d \\ &(c)\quad Z^x_{[}|_b|_{a]} + C_{[b}{}^j{}_{a]} Z^x{}_{|j} + C_{[b}{}^d{}_{a]} Z^x|_d = \\ &\qquad 2^{-1}(S_l{}^x{}_{ba} Z^l + S_d{}^x{}_{ba} Z^d) \qquad x \in \{k, c\} \end{aligned}$$

From (4.4) in [6] and (1.8) follows

Theorem 1.4. *The Ricci equations in the torsion free generalized Lagrange spaces are given by*

$$(1.9)\qquad \begin{aligned} &(a)\quad Z^x{}_{|j|i} - Z^x{}_{|i|j} = R_l{}^x{}_{ji} Z^l + R_d{}^x{}_{ji} Z^d \\ &(b)\quad Z^x{}_{|j}|_a - Z^x|_{a|j} = P_l{}^x{}_{ja} Z^l + P_d{}^x{}_{ja} Z^d \\ &(c)\quad Z^x|_b|_a - Z^x|_a|_b = S_l{}^x{}_{ba} Z^l + S_d{}^x{}_{ba} Z^d \\ &\qquad x \in \{k, c\}. \end{aligned}$$

It is clear that the connection coefficients which form the curvature tensors on the right hand side of equations (1.9) are also torsion free.

Theorem 1.5. *In the generalized torsion free Lagrange spaces the relations*

$$R_{l\ ji}^{\ k} = 0 \qquad R_{d\ ji}^{\ k} = 0$$

$$R_{l\ ji}^{\ c} = 0 \qquad R_{d\ ji}^{\ c} = 0$$

are valid, iff for every vector field $Z \in T(E)$ the equations

$$Z^{k}{}_{|j|i} = Z^{k}{}_{|i|j} \qquad Z^{c}{}_{|j|i} = Z^{c}{}_{|i|j}$$

are satisfied.

Theorem 1.6. *In the generalized torsion free Lagrange spaces the relations*

$$P_{l\ ja}^{\ k} = 0 \qquad P_{d\ ja}^{\ k} = 0$$

$$P_{l\ ja}^{\ c} = 0 \qquad P_{d\ ja}^{\ c} = 0$$

are valid, iff for every vector field $Z \in T(E)$ the equations

$$Z^{k}{}_{|j}|_{a} = Z^{k}|_{a|j} \qquad Z^{c}{}_{|j}|_{a} = Z^{c}|_{a|j}$$

are satisfied.

Theorem 1.7. *In the generalized torsion free Lagrange spaces the relations*

$$S_{l\ ba}^{\ k} = 0, \qquad S_{d\ ba}^{\ k} = 0$$

$$S_{l\ ba}^{\ c} = 0, \qquad S_{d\ ba}^{\ c} = 0$$

are valid, iff for every vector field $Z \in T(E)$ the equations

$$Z^{k}|_{b}|_{a} = Z^{k}|_{a}|_{b} \qquad Z^{c}|_{b}|_{a} = Z^{c}|_{a}|_{b}$$

are satisfied.

The Theorem 1.5.-1.7. are direct consequences of Theorem 1.4.

Theorem 1.8. *If in the generalized Lagrange space the connection coefficients satisfy the conditions*

(1.10) $\qquad F_{j\ i}^{\ c} = 0, \quad F_{a\ j}^{\ k} = 0, \quad C_{b\ a}^{\ k} = 0, \quad C_{j\ a}^{\ c} = 0,$

i.e. when (4.1) from [6] reduces to the form

$$\nabla_{\delta_i}\delta_j = F_{j\ i}^{\ k}\delta_k \qquad \nabla_{\delta_j}\partial_a = F_{a\ j}^{\ c}\partial_c$$
(1.11)
$$\nabla_{\partial_a}\delta_j = C_{j\ a}^{\ k}\delta_k \qquad \nabla_{\partial_a}\partial_b = C_{b\ a}^{\ c}\partial_c,$$

then the components of the curvature tensor satisfy the following equations

$$R_{d\ ji}^{\ k} = 0, \qquad R_{l\ ji}^{\ c} = 0$$
(1.12)
$$P_{d\ ja}^{\ k} = 0, \qquad P_{m\ ja}^{\ c} = 0$$
$$S_{d\ ba}^{\ k} = 0, \qquad S_{m\ ba}^{\ c} = 0.$$

The other components of the curvature tensors can be obtain from (1.3) if in them we substitute (1.10).

Theorem 1.9. *The Ricci equations for the torsion free Lagrange spaces supplied with Miron's d-connection (1.11) have the form:*

$$Z^k_{\ |j|i} - Z^k_{\ |i|j} = R_{l\ ji}^{\ k}Z^l \qquad Z^c_{\ |j|i} - Z^c_{\ |i|j} = R_{d\ ji}^{\ c}Z^d$$
(1.13)
$$Z^k_{\ |j}|_a - Z^k|_{a|j} = P_{l\ ja}^{\ k}Z^l \qquad Z^c_{\ |j}|_a - Z^c|_{a|j} = P_{d\ ja}^{\ c}Z^d$$
$$Z^k|_b|_a - Z^k|_a|_b = S_{l\ ba}^{\ k}Z^l \qquad Z^c|_b|_a - Z^c|_a|_b = S_{d\ ba}^{\ c}Z^d.$$

2. Ricci identities in generalized Hamilton spaces

In the tangent space $T(E)$ of the generalized Hamilton space $H^{n+m} = (E, M, N, G, \nabla, T\lambda)$ are given the vector fields

$$X = X^i\delta_i + X_a\partial^a \quad Y = Y^j\delta_j + Y_b\partial^b \quad Z = Z^k\delta_k + Z_c\partial^c. \qquad (*)$$

The curvature tensor R is defined as usually by

(2.1)
$$R = R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z = R^k\delta_k + R_c\partial^c$$

(*)The symbol *, which directs to the Hamilton space is omitted here.

Theorem 2.1. *The curvature tensor in the generalized Hamilton space has the form:*

$$(2.2)\quad \begin{aligned}(a)\quad R^k = {}& 2^{-1}(R_l{}^k{}_{ji}Z^l + R^{dk}{}_{ji}Z_d)(X^iY^j - Y^iX^j) + \\ & (P_l{}^k{}_j{}^a Z^l + P^{dk}{}_j{}^a Z_d)(X_aY^j - Y_aX^j) + \\ & 2^{-1}(S_l{}^{kba}Z^l + S^{dkba}Z_d)(X_aY_b - Y_aX_b)\\ (b)\quad R_c = {}& 2^{-1}(R_{lcji}Z^l + R^d{}_{cji}Z_d)(X^iY^j - Y^iX^j) + \\ & (P_{lcj}{}^a Z^l + P^d{}_{cj}{}^a Z_d)(X_aY^j - Y_aX^j) + \\ & 2^{-1}(S_{lc}{}^{ba}Z^l + S^d{}_c{}^{ba}Z_d)(X_aY_b - Y_aX_b),\end{aligned}$$

where

$$(2.3)\quad \begin{aligned}(a)\ 2^{-1}R_m{}^k{}_{ji} &= \delta_{[i}F_{|m|}{}^k{}_{j]} + F_m{}^h{}_{[j}F_{|h|}{}^k{}_{i]} + F_{mc[j}F^{ck}{}_{i]} \\ &\quad +2^{-1}C_m{}^{ke}K_{jei}\\ (b)\ 2^{-1}R^{dk}{}_{ji} &= \delta_{[i}F^{dk}{}_{j]} + F^d{}_{e[j}F^{ek}{}_{i]} + F^{dk}{}_{[j}F_{|h|}{}^k{}_{i]} + \\ &\quad +2^{-1}C^{dke}K_{jei}\\ (c)\ 2^{-1}R_{mcji} &= \delta_{[i}F_{|mc|j]} + F_{me[j}F^e_{|c|i]} + F_m{}^h{}_{[j}F_{|hc|i]} + \\ &\quad +2^{-1}C_{mc}{}^eK_{jei}\\ (d)\ 2^{-1}R^d{}_{cji} &= \delta_{[i}F^d{}_{|c|j]} + F^d{}_{e[j}F^e_{|c|i]} + F^{dh}{}_{[j}F_{|hc|i]} + \\ &\quad +2^{-1}C^d{}_c{}^eK_{jei}\\ (e)\ 2^{-1}K_{iej} &= \partial_{[j}N_{|e|i]} + N_{a[i}\partial^aN_{|e|j]},\quad K^a{}_{ej} = \partial^aN_{ej} - F^a{}_{ej}\end{aligned}$$

$$(2.4)\quad \begin{aligned}(a)\ P_m{}^k{}_j{}^a &= \partial^aF_m{}^k{}_j + K^a{}_{ej}C_m{}^{ke} - C_m{}^{ka}{}_{|j}\\ (b)\ P^{dk}{}_j{}^a &= \partial^aF^{dk}{}_j + K^a{}_{ej}C^{dke} - C^{dka}{}_{|j}\\ (c)\ P_{mcj}{}^a &= \partial^aF_{mcj} + K^a{}_{ej}C_{mc}{}^e - C_{mc}{}^a{}_{|j}\\ (d)\ P^d{}_{cj}{}^a &= \partial^aF^d{}_{cj} + K^a{}_{ej}C^d{}_c{}^e - C^d{}_c{}^a{}_{|j}\\ (e)\ C_m{}^{ka}{}_{|j} &= \delta_jC_m{}^{ka} + C_m{}^{ha}F_h{}^k{}_j + C_{me}{}^aF^{ek}{}_j - \\ &\quad C_h{}^{ka}F_m{}^h{}_j - C^{eka}F_{mej} - C_m{}^{ke}F^a{}_{ej}\end{aligned}$$

$$(2.5)\quad (a)\ 2^{-1}S_m{}^{kba} = \delta^{[a}C_m^{|k|b]} + C_m{}^{h[b}C_h{}^{|k|a]} + C_{me}{}^{[b}C^{|ek|a]}$$

$$
\begin{aligned}
(b)\ 2^{-1}S^{dkba} &= \partial^{[a}C^{|dk|b]} + C^{dk[b}C_h{}^{|k|a]} + C^d{}_e{}^{[b}C^{|eb|a]}\\
(c)\ 2^{-1}S_{mc}{}^{ba} &= \partial^{[a}C_{me}{}^{b]} + C_m{}^{h[b}C_{hc}{}^{a]} + C_{me}^{[b}C^{|e|}{}_c{}^{a]}\\
(d)\ 2^{-1}S^d{}_c{}^{ba} &= \partial^{[a}C^{|d|}{}_c{}^{b} + C^{dk[b}C_{hc}{}^{a]} + C^d{}_e{}^{[b}C^{|e|}{}_c{}^{a]}
\end{aligned}
$$

On the other hand starting from

$$
(2.6) \qquad \nabla_X Y = (Y^k_{|i}X^i + Y^k|^a X_a)\delta_k + (Y_{c|i}X^i + Y_c|^a X_a)\partial^c,
$$

where

$$
(2.7) \qquad
\begin{aligned}
(a)\quad & Y^k_{|i} = \delta_i Y^k + F_j{}^k{}_i Y^j + F^{bk}{}_i Y_b\\
(b)\quad & Y^k|^a = \partial^a Y^k + C_j{}^{ka}Y^j + C^{bka}Y_b\\
(c)\quad & Y_{c|i} = \delta_i Y_c + F_{jci}Y^j + F^b{}_{ci}Y_b\\
(d)\quad & Y_c|^a = \partial^a Y_c + C_{jc}{}^a Y^j + C^b{}_c{}^a Y_b
\end{aligned}
$$

we obtain:

Theorem 2.2. *The curvature tensor in the generalized Hamilton space defined by (2.1) has the form:*

$$
\begin{aligned}
(2.8) \qquad R(X,Y)Z &= (B^k{}_{ji}\delta_k + B_{cji}\partial^c)(X^iY^j - Y^iX^j) +\\
&\quad (B^k{}_j{}^a\delta_k + B_{cj}{}^a\partial^c)(X_aY^j - Y_aX^j) +\\
&\quad (B^{kba}\delta_k + B_c{}^{ba}\partial^c)(X_aY_b - Y_aX_b),
\end{aligned}
$$

where

$$
\begin{aligned}
(2.9)\quad B^k{}_{ji} &= Z^k_{|[j|i]} + F_{[j}{}^h{}_{i]}Z^k{}_{|h} + (F_{[j|b|i]} + 2^{-1}K_{jbi})Z^k|^b\\
B_{cji} &= Z_{c|[j|i]} + F_{[j}{}^h{}_{i]}Z_{c|h} + (F_{[j|b|i]} + 2^{-1}K_{jbi})Z_c|^b\\
B^k{}_j{}^a &= Z^k{}_{|j}|^a - Z^k|^a{}_{|j} + (C_j{}^{ha} - F^{ah}{}_j)Z^k{}_{|h} + (K^a{}_{bj} + C_{jb}{}^a)Z^k|^b\\
B_{cj}{}^a &= Z_{c|j}|^a - Z_c|^a{}_{|j} + (C_j{}^{ha} - F^{ah}{}_j)Z_{c|h} + (K^a{}_{bj} + C_{jb}{}^a)Z_c|^b\\
B^{kba} &= Z^k|^{[b}|^{a]} + C^{[b|j|a]}Z^k{}_{|j} + C^{[b}{}_d{}^{a]}Z^k|^d\\
B_c{}^{ba} &= Z_c|^{[b}|^{a]} + C^{[b|j|a]}Z_{c|j} + C^{[b}{}_d{}^{a]}Z_c|^d.
\end{aligned}
$$

Comparing (2.2) with (2.8) and (2.9) we get

Theorem 2.3. *The Ricci equations for the generalized Hamilton spaces are:*

$$(2.10)\quad (a)\quad Z^k_{|[j|i]} + F_{[j}{}^h{}_{i]} Z^k{}_{|h} + (F_{[j|b|i]} + 2^{-1}K_{jbi}) Z^k|^b = 2^{-1}(R_l{}^k{}_{ji} Z^l + R^{dk}{}_{ji} Z_d)$$

$$(b)\quad Z^k{}_{|j}|^a - Z^k|^a_{|j} + (C_j{}^{ha} - F^{ah}{}_j) Z^k{}_{|h} + (K^a{}_{bj} + C_{jb}{}^a) Z^k|^b = P_l{}^k{}_j{}^a Z^l + P^{dk}{}_j{}^a Z_d$$

$$(c)\quad Z^k|^{[b}|^{a]} + C^{[b|j|a]} Z^k_{|j} + C^{[b}{}_d{}^{a]} Z^k|^d = 2^{-1}(S_l{}^{kba} Z^l + S^{dkba} Z_d)$$

$$(d)\quad Z_{c|[j|i]} + F_{[j}{}^h{}_{i]} Z_{c|h} + (F_{[j|b|i]} + 2^{-1}K_{jbi}) Z_c|^b = 2^{-1}(R_{lcji} Z^l + R^d{}_{cji} Z_d)$$

$$(e)\quad Z_{c|j}|^a - Z_c|^a{}_{|j} + (C_j{}^{ha} - F^{ah}{}_j) Z_{c|h} + (K^a{}_{bj} + C_{jb}{}^a) Z_c|^b = P_{lcj}{}^a Z^l + P^d{}_{cj}{}^a Z_d$$

$$(f)\quad Z_c|^{[b}|^{a]} + C^{[b|j|a]} Z_{c|j} + C^{[b}{}_d{}^{a]} Z_c|^d = 2^{-1}(S_{lc}{}^{ba} Z^l + S^d{}_c{}^{ba} Z_d)$$

From (4.4) in [6] and (2.10) follows

Theorem 2.4. *The Ricci equtations in the torsion free generalized Hamilton spaces are given by*

$$(2.11)\quad (a)\quad Z^k{}_{|j|i} - Z^k{}_{|i|j} = R_l{}^k{}_{ji} Z^l + R^{dk}{}_{ji} Z_d$$

$$(b)\quad Z^k{}_{|j}|^a - Z^k|^a{}_{|j} = P_l{}^k{}_j{}^a Z^l + P^{dk}{}_j{}^a Z_d$$

$$(c)\quad Z^k|^b|^a - Z^k|^a|^b = S_l{}^{kba} Z^l + S^{dkba} Z_d$$

$$(d)\quad Z_{c|j|i} - Z_{c|i|j} = R_{lcji} Z^l + R^d{}_{cji} Z_d$$

$$(e)\quad Z_{c|j}|^a - Z^c|^a{}_{|j} = P_{lcj}{}^a Z^l + P^d{}_{cj}{}^a Z_d$$

$$(f)\quad Z_c|^b|^a - Z_c|^a|^b = S_{lc}{}^{ba} Z^l + S^d{}_c{}^{ba} Z_d.$$

Remark. The connection coefficients which form the curvature tensors on the right hand side of equations (2.11) are also torsion free.

Theorem 2.5. *In the genelarized torsion free Hamilton spaces the relations*

$$R_l{}^k{}_{ji}=0,\ \ R^{dk}{}_{ji}=0,\ \ R_{lcji}=0,\ \ R^d{}_{cji}=0$$

are valid, iff for every vector field $Z\in T(E^*)$ the equations

$$Z^k{}_{|j|i}=Z^k{}_{|i|j}\qquad\qquad Z_{c|j|i}=Z_{c|i|j}$$

are satisfied.

Theorem 2.6. *In the generalized torsion free Hamilton spaces the relations*

$$P_l{}^k{}_j{}^a=0,\ \ P^{dk}{}_j{}^a=0,\ \ P_{lcj}{}^a=0,\ \ P^d{}_{cj}{}^a=0$$

are valid iff for every vector field $Z\in T(E^*)$ the equations

$$Z^k_{|j}|^a=Z^k|^a{}_{|j},\qquad\qquad Z_{c|j}|^a=Z_c|^a{}_{|j}$$

are satisfied.

Theorem 2.7. *In the generalized torsion free Hamilton spaces the relations*

$$S_l{}^{kba}=0,\ \ S^{dkba}=0,\ \ S_{lc}{}^{ba}=0,\ \ S^d{}_c{}^{ba}=0$$

are valid iff for every vector field $Z\in T(E^*)$ the equations

$$Z^k|^b|^a=Z^k|^a|^b\quad Z_c|^b|^a=Z_c|^a|^b$$

are satisfied.

The Theorems 2.5.-2.7. are direct consequences of Theorem 2.4.

Theorem 2.8. *If in the generalized Hamilton space the connection coefficients satisfy the conditions*

(2.12) $$F_{jci}=0,\ \ F^{ah}{}_j=0,\ \ C_{jc}{}^a=0,\ \ C^{bka}=0,$$

i.e. when (4.1) from [6] reduces to the form

(2.13) $$\begin{aligned}&\nabla_{\delta_i}\delta_j=F_j{}^k{}_i\delta_k &&\nabla_{\delta_j}\partial^a=F^a{}_{cj}\partial^c\\ &\nabla_{\partial^a}\delta_j=C_j{}^{ka}\delta_k &&\nabla_{\partial^a}\partial^b=C^b{}_c{}^a\partial^c,\end{aligned}$$

then the components of the curvature tensor satisfy the following equations:

$$
\text{(2.14)} \qquad
\begin{aligned}
R^{dk}{}_{ji} &= 0 \qquad & R_{hcji} &= 0 \\
P^{dk}{}_{j}{}^{a} &= 0 \qquad & P_{mcj}{}^{a} &= 0 \\
S^{dkba} &= 0 \qquad & S_{mc}{}^{ba} &= 0.
\end{aligned}
$$

The other components of the curvature tensor can be obtained from (2.3), (2.4) and (2.5) if in them we substitute (2.12).

Theorem 2.9. *The Ricci equations for the torsion free Hamilton spaces supplied with Miron's d-connection (2.13) have the form:*

$$
\text{(2.15)} \qquad
\begin{aligned}
Z^{k}{}_{|j|i} - Z^{k}_{|i|j} &= R_{l}{}^{k}{}_{ji} Z^{l} \qquad & Z_{c|j|i} - Z_{c|i|j} &= R^{d}{}_{cji} Z_{d} \\
Z^{k}{}_{|j}|^{a} - Z^{k}|^{a}{}_{|j} &= P_{l}{}^{k}{}_{j}{}^{a} Z^{l} \qquad & Z_{c|j}|^{a} - Z^{c}|^{a}_{|j} &= P^{d}{}_{cj}{}^{a} Z_{d} \\
Z^{k}|^{b}|^{a} - Z^{k}|^{a}|^{b} &= S_{l}{}^{kba} Z^{l} \qquad & Z_{c}|^{b}|^{a} - Z^{c}|^{a}|^{b} &= S^{d}{}_{c}{}^{ba} Z_{d}.
\end{aligned}
$$

3. The Bianchi identities in the generalized Lagrange spaces

Let us denote as usually:

$$\text{(3.1)} \qquad (\nabla_X T)(Y,Z) = \nabla_X (T(Y,Z)) - T(\nabla_X Y, Z) - T(Y, \nabla_X Z)$$

$$\text{(3.2)} \qquad \begin{aligned}(\nabla_X R)(Y,Z,U) = \nabla_X (R(Y,Z,U)) - R(\nabla_X Y, Z, U) - \\ - R(Y, \nabla_X Z, U) - R(Y, Z, \nabla_X U),\end{aligned}$$

where X, Y, Z, U are vector fields in $T(E)$, then form (1.1) and

$$\text{(3.3)} \qquad T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

we obtain

Theorem 3.1. *The Bianchi identities for the generalized Lagrange spaces have the form:*

(3.4) $\underset{XYZ}{\sigma} \{(\nabla_X R)(Y,Z,U) + R(T(X,Y),Z)U\} = 0$

(3.5) $\underset{XZY}{\sigma} \{(\nabla_X T)(Y,Z) - R(X,Y)Z - T(T(X,Y),Z)\} = 0$

The proof is obtained from (3.1)-(3.3).

Theorem 3.2. *The explicit from of Bianchi identities for the generalized Lagrange spaces are:*

$$\begin{aligned}
(3.6)(a)\quad & 2^{-1}(R_l{}^x{}_{ji} Z^l + R_d{}^x{}_{ji} Z^d)_{|m} = A^x{}_{ji|m} \\
(b)\quad & 2^{-1}(R_l{}^x{}_{ij} Z^l + R_d{}^x{}_{ij} Z^d)|_a + 2^{-1}[(P_l{}^x{}_{ja} Z^l + P_d{}^x{}_{ja} Z^d)_{|i} - (i,j)] \\
& = A^x{}_{ij}|_a + 2^{-1}(A^x{}_{ja|i} - A^x{}_{ia|j}) \\
(c)\quad & 2^{-1}(S_l{}^x{}_{ba} Z^l + S_d{}^x{}_{ba} Z^d)_{|m} - 2^{-1}[(P_l{}^x{}_{ma} Z^l + P_d{}^x{}_{ma} Z^d)|_b - (a,b)] \\
& = A^x{}_{ba|m} - 2^{-1}(A^x{}_{ma}|_b - A^x{}_{mb}|_a) \\
(d)\quad & 2^{-1}(S_l{}^x{}_{ba} Z^l + S_d{}^x{}_{ba} Z^d)|_e = A^x{}_{ba}|_e
\end{aligned}$$

where the tensor A is defined by (1.7) and $x \in \{c,k\}$.

Theorem 3.3. *The explicit form of Bianchi identities for the torsion free generalized Lagrange spaces are:*

$$\begin{aligned}
(3.7)\qquad (a)\quad & (R_l{}^x{}_{ji} Z^l + R_d{}^x{}_{ji} Z^d)_{|m} = (Z^x{}_{|j|i} - Z^x{}_{|i|j})_{|m} \\
(b)\quad & (R_l{}^x{}_{ij} Z^l R_d{}^x{}_{ij} Z^d)|_a + (P_l{}^x{}_{ja} Z^l + P_d{}^x{}_{ja} Z^d)_{|i} - \\
& -(P_l{}^x{}_{ia} Z^l + P_d{}^x{}_{ia} Z^d)_{|j} = \\
& = [(Z^x{}_{|j}|_{a|i} + Z^x|_{a|i|j} + Z^x{}_{|i|j}|_a) - (i,j)] \\
(c)\quad & (S_l{}^x{}_{ba} Z^l + S_d{}^x{}_{ba} Z^d)_{|m} - (P_l{}^x{}_{ma} Z^l + P_d{}^x{}_{ma} Z^d)|_b + \\
& +(P_l{}^x{}_{mb} Z^l + P_d{}^x{}_{mb} Z^d)|_a = \\
& [(Z^x|_b|_{a|m} + Z^x|_{a|m}|_b + Z^x{}_{|m}|_b|_a) - (a,b)] \\
(d)\quad & (S_l{}^x{}_{ba} Z^l + S_d{}^x{}_{ba} Z^d)|_e = Z^x|_b|_a|_e - Z^x|_a|_b|_e \\
& x \in \{c,k\}
\end{aligned}$$

The Bianchi identity (3.5) for tensor $R_l{}^x{}_{ji}$ in the generalized Lagrange space has the form

(3.8) $\underset{lji}{\sigma} \{R_l{}^k{}_{ji} - T_l{}^k{}_{j|i} - C_l{}^k{}_b K_j{}^b{}_i + F_e{}^k{}_i \delta_l N_j^e - F_e{}^k{}_i \delta_j N_l^e\} = 0$

If the distribution N in the generalized Lagrange space is integrable, then we obtain

$$\underset{lji}{\sigma}\{R_{l\ ji}^{\ k} - T_{l\ j|i}^{\ k} - C_{l\ b}^{\ k}K_{j\ i}^{\ b}\} = 0.$$

From Bianchi identity (3.5) for the tensor $S_{c\ ba}^{\ d}$ in (1.3) relation

$$\underset{cba}{\sigma}\{S_{c\ ba}^{\ d} + T_{b\ c}^{\ d}|_a + C_{f\ a}^{\ d}T_{b\ c}^{\ f}\} = 0 \tag{3.9}$$

holds good.

From (1.3) and using [6] for the tensor $P_{k\ ja}^{\ l}$ we obtain:

$$\begin{aligned} P_{k\ ja}^{\ l} - P_{j\ ka}^{\ l} - T_{k\ j}^{\ l}|_a + C_{i\ a}^{\ l}T_{k\ j}^{\ i} - C_{k\ a}^{\ i}T_{j\ i}^{\ l} - \\ -C_{j\ a|k}^{\ l} + C_{k\ a|j}^{\ l} - K_{a\ j}^{\ b}C_{k\ b}^{\ l} + K_{a\ k}^{\ b}C_{j\ b}^{\ l} = 0. \end{aligned} \tag{3.10}$$

4. Bianchi identites in generalized Hamilton spaces

Since relations (3.1)-(3.3) are valid for every vector field $X, Y, Z, U \in T(E)$ it follows that the formulae (3.4) and (3.5) are also satisfied in generalized Hamilton spaces.

Theorem 4.1. *The explicit from of Bianchi identites for the generalized Hamilton spaces are:*

(4.1)

$$\begin{aligned}
&(a)\quad 2^{-1}(R_{h\ ji}^{\ k}Z^h + R^{dk}_{\ \ ji}Z_d)_{|m} = B^k_{\ ji|m}\\
&(b)\quad 2^{-1}(R_{hcji}Z^h + R^d_{\ cji}Z_d)_{|m} = B_{cji|m}\\
&(c)\quad 2^{-1}(R_{h\ ij}^{\ k}Z^h + R^{dk}_{\ \ ij}Z_d)|^a +\\
&\qquad +2^{-1}[(P_{h\ j}^{\ k\ a}Z^h + P^{dk\ a}_{\ \ j}Z_d)_{|i} - (i,j)] =\\
&\qquad = B^k_{\ ij}|^a + 2^{-1}(B^{k\ a}_{\ j\ |i} - B^{k\ a}_{\ i\ |j})\\
&(d)\quad 2^{-1}(R_{hcij}Z^h + R^d_{\ cij}Z_d)|^a +\\
&\qquad 2^{-1}[(P_{hcj}^{\ \ \ a}Z^k + P^d_{\ cj}{}^aZ_d)_{|i} - (i,j)] =\\
&\qquad = B_{cji}|^a + 2^{-1}(B_{cj\ |i}^{\ \ a} - B_{ci\ |j}^{\ \ a})\\
&(e)\quad 2^{-1}(S_h^{\ kba}Z^h + S^{dhba}Z_d)_{|i} -\\
&\qquad 2^{-1}[(P_{h\ i}^{\ k\ a}Z^h + P^{dk\ a}_{\ \ i}Z_d)|^b - (a,b)] =
\end{aligned}$$

$$B^{kba}{}_{|i} - 2^{-1}(B^{k}{}_{i}{}^{a}|^{b} - B^{k}{}_{i}{}^{b}|^{a})$$

$$(f)\quad 2^{-1}(S_{hc}{}^{ba}Z^{h} + S^{d}{}_{c}{}^{ba}Z_{d})_{|i} - 2^{-1}[(P_{hci}{}^{a}Z^{h} + P^{d}{}_{ci}{}^{a}Z_{d})|^{b} - (a,b)] = B_{c}{}^{ba}{}_{|i} - 2^{-1}(B_{ci}{}^{a}|^{b} - B_{ci}{}^{b}|^{a})$$

$$(g)\quad 2^{-1}(S_{h}{}^{kba}Z^{h} + S^{dkba}Z_{d})|^{e} = B^{kba}|^{e}$$

$$(h)\quad 2^{-1}(S_{hc}{}^{ba}Z^{h} + S^{d}{}_{c}{}^{ba}Z_{d})|^{e} = B_{c}{}^{ba}|^{e}.$$

where the components of the tensor B are determined by (2.9).

Theorem 4.2. *In the torsion free generalized Hamilton spaces supplied with Miron's d-connection (2.13) the Bianchi identitites have the form:*

(4.2)

$$(a)\quad (R_{h}{}^{k}{}_{ji}Z^{h})_{|m} = Z^{h}{}_{|j|i|m} - Z^{h}{}_{|i|j|m}$$

$$(b)\quad (R^{d}{}_{cji}Z_{d})_{|m} = Z_{c|j|i|m} - Z_{c|i|j|m}$$

$$(c)\quad (R_{h}{}^{k}{}_{ij}Z^{h})|^{a} + (P_{h}{}^{k}{}_{j}{}^{a}Z^{h})_{|i} - (P_{h}{}^{k}{}_{i}{}^{a}Z^{h})_{|j} = [(Z^{k}{}_{|j}|^{a}{}_{|i} + Z^{k}{}_{|i|j}|^{a} + Z^{k}|^{a}{}_{|i|j}) - (i,j)]$$

$$(d)\quad (R^{d}{}_{cji}Z_{d})|^{a} + (P^{d}{}_{cj}{}^{a}Z_{d})_{|i} - (P^{d}{}_{ci}{}^{a}Z_{d})_{|j} = [(Z_{c|j}|^{a}{}_{|i} + Z_{c|i|j}|^{a} + Z_{c}|^{a}{}_{|i|j}) - (i,j)]$$

$$(e)\quad (S_{h}{}^{kba}Z^{h})_{|i} - (P_{h}{}^{k}{}_{i}{}^{a}Z^{h})|^{b} + (P_{h}{}^{k}{}_{i}{}^{b}Z^{h})|^{a} = [(Z^{k}|^{b}|^{a}{}_{|i} + Z^{k}|^{a}{}_{|i}|^{b} + Z^{h}{}_{|i}|^{b}|^{a}) - (a,b)]$$

$$(f)\quad (S^{d}{}_{c}{}^{ba}Z_{d})_{|i} - (P^{d}{}_{ci}{}^{a}Z_{d})|^{b} + (P^{d}{}_{ci}{}^{b}Z_{d})|^{a} = [(Z_{c}|^{b}|^{a}{}_{|i} + Z_{c}|^{a}{}_{|i}|^{b} + Z_{c|i}|^{b}|^{a}) - (a,b)]$$

$$(g)\quad (S_{h}{}^{kba}Z^{h})|^{e} = Z^{k}|^{b}|^{a}|^{e} - Z^{k}|^{a}|^{b}|^{e}$$

$$(h)\quad (S^{d}{}_{c}{}^{ba}Z_{d})|^{e} = Z_{c}|^{b}|^{a}|^{e} - Z_{c}|^{a}|^{b}|^{e}$$

The proof follows from (4.1) and (4.4) in [6].

Using the comparative characterisations of generalized Lagrange and generalized Hamilton spaces the corresponding equations to (3.8), (3.9), (3.10) for the generalized Hamilton spaces have the forms as in (4.3), (4.4) and (4.5):

(4.3)

$$\overset{\sigma}{\scriptstyle lji}\{R_{l}{}^{k}{}_{ji} - T_{l}{}^{k}{}_{j|i} - C_{l}{}^{ke}K_{jei} + F^{ek}{}_{i}\delta_{l}N_{ej} - F^{ek}{}_{i}\delta_{j}N_{el}\} = 0.$$

If the distribution N in the generalized Lagrange space is integrable, then we obtain

$$\underset{lji}{\overset{\sigma}{}}\{R_l{}^k{}_{ji} - T_l{}^k{}_{j|i} - C_l{}^{ke}K_{jei}\} = 0.$$

(4.4)
$$\underset{cba}{\overset{\sigma}{}}\{S^c{}_d{}^{ba} + T^b{}_d{}^c|_a C^f{}_d{}^a T^b{}_f{}^c\} = 0$$

(4.5)
$$P_k{}^l{}_j{}^a - P_j{}^l{}_k{}^a - T_k{}^l{}_j|^a + C_i{}^{la}T_k{}^i{}_j -$$

$$C_k{}^{ia}T_j{}^l{}_i - C_j{}^{la}{}_{|k} - C_k{}^{la}{}_{|j} - K^a{}_{bj}C_k{}^{lb} + K^a{}_{bk}C_j{}^{lb} = 0$$

and for $P_{klj}{}^a$ we have on the analogous way:

$$P_{klj}{}^a - P_{jlk}{}^a - T_{klj}|_a - C_{il}{}^a T_k{}^i{}_j -$$

$$-C_k{}^{ia}T_{jli} - C_{jl}{}^a{}_{|k} + C_{kl}{}^a{}_{|j} - K^a{}_{bj}C_{kl}{}^b K^a{}_{bk}C_{jl}{}^b = 0.$$

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REZIME

KOMPARATIVNA KARAKTERIZACIJA GENERALISANIH LAGRANGEOVIH I GENERALISANIH HAMILTONOVIH PROSTORA

Ovaj rad je nastavak rada [6]. Koristiće se oznake i definicije date u njoj. U [6] su uvedeni generalisani Lagrangeovi i Hamiltonovi prostori i odredjeni su koeficijenti koneksije i torzije za rekurentne prostore. Ovde će za ove prostore biti data teorija krivina kao i Riccieve i Bianchieve jednačine.

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