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A NOTE ON WEAKLY AND ALMOST CONTINUOUS MULTIFUNCTIONS

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Abstract

In [5] and [6] Kovačević introduced the notions of an α-regular set and α-almost regular set. The purpose of the present note is to generalize some results from [7] - [11], [13] and [16], using the notions of an α-regular set and α-almost regular set.

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1. Introduction

The weak continuous multifunctions are defined by us in [9] as generalizations of univocal weak continuous functions defined by Levine in [7]. Smithson also defines weak continuous multifunctions in [17]. Some of the applications of weak continuous multifunctions are given in [1], [2], [10], [13], [14].

In [11] we defined almost continuous multifunctions as a generalization of univocal almost continuous functions defined by Singal and Singal in [16]. The properties of almost continuous multifunctions are studied in [4], [12] and [13].

Definition 1. *Let X and Y be two topological spaces.*

a) *The multifunction $F : X \to Y$ is upper weakly continuous (u. w. c.), respectively, upper almost continuos (u. a. c.), in the point $x \in X$, if for every open set $G \subset Y$ with $F(x) \subset G$, there exists an open set $V \subset X$ containing x, so that $F(V) \subset Cl(G)$, respectively, $F(V) \subset Int\, Cl(G)$ ($Cl(G)$ stands for the closure of G).*

b) *The multifunction $F : X \to Y$ is lower weakly continuous (l. w. c.), respectively, lower almost continuous (l. a. c.) in the point $x \in X$, if for every open set $G \subset Y$ with $F(x) \cap G \neq \emptyset$, there exists an open set $V \subset Y$ containing x, so that $F(y) \cap Cl(G)$, $\forall y \in V$, respectively, $F(y) \cap\ IntCl(G) \neq \emptyset$, $\forall y \in V$.*

c) *The multifunction $F : X \to Y$ is weakly continuous (w. c.), respectively, almost continuous (a. c.) in the point $x \in X$, if it is upper and lower weakly continuous, respectively, upper and lower almost continuous in the point x.*

d) *The multifunction $F : X \to Y$ is weakly continuous (u. w. c., l. w. c.), respectively, almost continuous (u. a. c., l. a. c.), if it has this property in any point $x \in X$ [9], [11], [17].*

The following implication holds:

(1) u.s.c. $\Rightarrow$ u.a.c.$\Rightarrow$ u.w.c.

(2) l.s.c. $\Rightarrow$ l.a.c. $\Rightarrow$ l.w.c. [9], [11], [17].

In the present paper using the notions of an α-regular set and α- almost regular set defined by Kovačević in [5] and [6] we shall generalize some results from [7] – [11], [13] and [16].

Definition 2. *Let X be a topological space and A a subset of X. The set A is α- regular (α- almost regular) if for any point $a \in A$ and any open (regulary open) set U containing a, there exists an open set V such that $a \in V \wedge Cl(V) \subset U$ [5], [6].*

Definition 3. *A subset A of a topological space X is strictly paracompact [3] (or α-paracompact [18]), if for every X-open cover U of A there exists on X-open X-locally finite family V which refines U and cover A.*

Lemma 1. *If A is an α-regular α-paracompact subset of a space X, U an open neighbourhoud of A, then there exists an open neighbourhoud V of A such that $A \subset V \subset Cl(V) \subset U$. ([5], Theorem 2.5).*

Lemma 2. *If A is an α-almost regular α-paracompact subset of a topological space X, U a regular open neighbourhood of A, then there exists an open neighbourhood V of A such that $A \subset V \subset Cl(V) \subset U$.*

Proof. It is similar to the proof of Theorem 2.5 of [5].

2. Results

Theorem 1. *For the multifunction $F : X \to Y$ for which $F(x)$ is an α-regular α-paracompact set of Y the concept of multifunction u. w. c. in $x \in X$ and the concept of multifunction u. s. c. in $x \in X$ coincide.*

Proof. Let us suppose the multifunction F u. w. c. in x. Let G be an open set such that $F(x) \subset G$. As $F(x)$ is α-regular α-paracompact by Lemma 1 there exists an open set V such that $F(x) \subset V \subset Cl(V) \subset G$. Since F is u. w. c. in x and $F(x) \subset V$ there exists an open set $U \subset X$ such that $x \in U$ and $F(U) \subset Cl(V)$ and thus $F(U) \subset G$. The multifunction F is thus u. s. c.

The reciprocity is obvious.

Corollary 1. *For the multifunction $F : X \to Y$ where Y is a regular space and for which $F(x)$ has a finite number of elements, the concept of multifunction u. w. c. in x coincides with the concept of multifunction u. s. c. in x ([9], Theorem 1).*

Corollary 2. *For the multifunction $F : X \to Y$ with Y T_3 space and for which $F(x)$ is strictly paracompact set, the concept of multifunction u. w. c. in x coincides with the concept of multifunction u. s. c. in x ([11], Theorem 3).*

Lemma 3. *If A is an α-regular set of X, then for every open set D which intersects A, there exists an open set D_A so that $A \cap D_A \neq \emptyset$ and $Cl(D_A) \subset D$.*

Proof. Let A be an α-regular subset of X and D an open set such that $A \cap D \neq \emptyset$. Let $x \in A \cap D$. Then $x \bar{\in} (X - D)$ and $(X - D)$ is a closed set. By Definition 2.2 of [5] there exist disjoint open neighbourhoods of x and $(X - D)$. Thus $x \in U$, $X - D \subset V$ and $U \cap V = \emptyset$, which implies $U \cap Cl(V) = \emptyset$ and $x \bar{\in} Cl(V)$. Let $D_A = X - Cl(V)$. Then $x \in D_A$ and thus $A \cap D_A \neq \emptyset$. On other hand, $Cl(D_A) = Cl(X - Cl(V)) \subset Cl(X - V) = X - V \subset D$.

Theorem 2. *For the multifunction $F : X \to Y$ for which $F(x)$ is an α-regular set, the concept of multifunction l. w. c. in x coincides with the concept of multifunction l. s. c. in x.*

Proof. Let us suppose F l. w. c. in x and G an open set of Y such that $F(x) \cap G \neq \emptyset$. As $F(x)$ is α-regular, by Lemma 3 there exists an open set $D \subset Y$ such that $F(x) \cap D \neq \emptyset$ and $Cl(D) \subset G$. As F is l. w. c. in x, there exists an open set $V \subset X$ with $x \in V$ and $F(y) \cap Cl(D) \neq \emptyset$, $\forall y \in V$ and thus $F(y) \cap G \neq \emptyset$, $\forall y \in V$ as well, and this shows that F is l. s. c. in x.

Reciprocally, if the multifunction F is l. s. c. in x it is obvious by l. w. c. in x.

Corollary 3. *For the multifunction $F : X \to Y$, where Y is a regular space, the l. w. c. and l. s. c. multifunctions notions coincide. ([9] , Theorem 2).*

Corollary 4. *For the single valued multifunction $F : X \to Y$, with a Y regular space, the notions of weak continuity and continuity coincide. ([7], Theorem 2).*

Definition 4. *A topological space X is almost regular if for any point $x \in X$ and each regular closed $A \subset X$ with $x \bar{\in} A$, there exist disjoint open sets U and V such that $x \in U$ and $A \subset V$ [15].*

Theorem 3. *For the multifunction $F : X \to Y$, for which $F(x)$ is an α-almost regular α-paracompact subset of Y the concept of multifunction u. w. c. in x and the concept of multifunction u. a. c. in x coincide.*

Proof. Let G be a regular open set in Y and $x \in F^+(G)$, namely $F(x) \subset G$. Since $F(x)$ is α-almost regular by Lemma 2 there exists an open set $V \subset Y$ such that $F(x) \subset V \subset Cl(V) \subset G$. Since F is u. w. c. and $F(x) \subset V$, there exists an open set $U \subset X$ such that $x \in U$ and $F(U) \subset Cl(V) \subset G$. So, $x \in U \subset F^+(G)$. Thus $F^+(G)$ is open in X and by Theorem 2.4, the implication (3) $\Rightarrow$ (1) of [11] F is u. a. c.

Reciprocally, if the multifunction F is u. a. c. in x, it is obvoius u. w. c. in x, as well.

Corollary 5. *For the multifunction $F : X \to Y$, where Y is an almost regular space, and for which $F(x)$ is a strictly paracompact set, the concept of multifunction u. w. c. in x coincides with the concept of multifunction u. a. c. in x ([13], Theorem 2.3).*

Lemma 4. *If A is an α-almost regular set of X, then for any regular closed set F of X and any point $x \in A$ such that $x \in X - F$, there exist disjoint open sets containing x and F, respectively.*

Proof. Let $x \in A$ and $x \bar{\in} F$. Since $x \bar{\in} F$, then $x \in X - F$. As $X - F$ is a regular open set and $x \in X - F$ there is an open set V such that $x \in V \subset Cl(V) \subset X - F$. Let $D = X - Cl(V)$, then $D \supset F, x \in V$ and $D \cap V \neq \emptyset$.

Lemma 5. *If A is an α-almost regular set of X, then for every regular open set D which intersects A, there exists an open set D_A, so that $A \cap D_A \neq \emptyset$ and $Cl(D_A) \subset D$.*

Proof. It is similar to the proof of Lemma 3 and follows by Lemma 4 and Definition 2.

Theorem 4. *For the multifunction $F : X \to Y$, for which $F(x)$ is an α-almost regular set, the concept of multifunction l. w. c. in x coincides with the concept of multifunction l. a. c. in x.*

Proof. Let us suppose that F l. w. c. in x and G is a regular open set of Y such that $F(x) \cap G \neq \emptyset$. Since $F(x)$ is α-almost regular by Lemma 5 there exists an open set $D \subset Y$, as $F(x) \cap D \neq \emptyset$ and $Cl(D) \subset G$. Since F is l. w. c. in x, there exists an open set $V \subset X$ with $x \in V$ and $F(y) \cap Cl(D) \neq \emptyset$, $\forall y \in V$ and thus $F(y) \cap G \neq \emptyset$, $\forall x \in V$. So, $x \in V \subset F^{-}(G)$ which shows that $F^{-}(G)$ is open, and so F is l. a. c. according to Theorem 2.2, implication (3) $\Rightarrow$ (1) of [11].

Corollary 6. *For the multifunction $F : X \to Y$, where Y is an almost regular space, weak continuity coincides with the almost continuity ([8], Theorem 1).*

Corollary 7. *For univocal applications with values in an almost regular space, weak continuity coincides with the almost continuity ([8], Theorem 1).*

Definition 5. *Let X be a topological space and A a subset of X. The set A is α-semiregular if for any point $a \in A$ and any open set U containing a, there exists a regular open set V such that $x \in V \subset U$.*

Lemma 6. *If A is an α-semiregular set of X, then for every open set D which intersects A, there is a regular open set D_A, such that $A \cap D_A \neq \emptyset$ and $D_A \subset D$.*

Proof. It is similar to the proof of Lemma 3.1 of [11].

Theorem 5. *For the multifunction $F : X \to Y$, for which $F(x)$ is an α-semiregular set of Y, the concept of l. a. c. multifunction in x and the concept of u. s. c. multifunction in x coincide.*

Proof. Let us suppose the multifunction $F : X \to Y$ l. a. c. in x. Let G be an open set, such that $F(x) \cap G \neq \emptyset$. Since $F(x)$ is α-semiregular by Lemma 6, there exists a regular open set $D \subset Y$ as $F(x) \cap D \neq \emptyset$ and $D \subset G$. As F is l. a. c. in x, from Theorem 2.1, implication (1) $\Rightarrow$ (4) of [11] results that there is an open set $U \subset X$ such that $x \in U$ and $F(x) \cap D \neq \emptyset$, $\forall x \in U$, so $F(x) \cap G \neq \emptyset$, $\forall x \in U$ and thus shows that F is u. s. c. in x.

Reciprocally, if the multifunction F is u. s. c. in x, it is obviously l. a. c. in x as well.

Corollary 8. *For the multifunction $F : X \to Y$, where Y is a semiregular space, the notions l. s. c. and l. a. c. coincide ([11], Theorem 3.1).*

Corollary 9. *For single valued mapping $f : X \to Y$ with a Y semiregular space, the notion of almost continuous function and continuous function coincide ([16], Theorem 2.4).*

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REZIME

PRIMEDBE NA SLABO I SKORO NEPREKIDNE MULTIFUNKCIJE

U [5] i [6] Kovačević je uveo pojam α-regularnih i α-skoro regularnih skupova. Cilj ove primedbe je da se generališu rezultati dobijeni u [7]-[11], [13] i [16], koristeći pojmove α-regularnih i skoro regularnih skupova.

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