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THE NUCLEUS OF A LATTICE [1]

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Abstract

The set of all the reducible elements of a lattice L, denoted by $K(L)$ is considered, and its properties in connection with sublattices, morphisms and products. The class $\mathcal{K}$ of all the lattices such that $K(L) = L$ is not an equational class. Every lattice can be embedded in some element of $\mathcal{K}$.

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1. Introduction

It is well known that the join-irreducible elements play an important role in lattices. For finite lattices they make one generating set. Also, the set of nonzero join-irreducible elements plays the main role in the proof of the Representation theorem, and makes the class of all the distributive lattices correspond to the class of all the finite posets.

Here we shall observe the set of all the irreducible (that is join- and meet-irreducible) elements and its complement, the nucleus of a lattice.

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Definition 1. *The nucleus of the lattice* $(L, \wedge, \vee)$ *is the set:*

$$K(L) = \{x \in L | (\exists y, z \in L)(y \| z \text{ and } (x = y \wedge z \text{ or } x = y \vee z)),$$

where $x \| y$ *means that* x *and* y *are incomparable.*

Theorem 1. *If* $K(L) \neq \emptyset$, *then* $K(L)$ *is a sublattice of* L.

Proof. Let $x, y \in K(L)$. If x and y are incomparable, by Definition 1 $x \wedge y$, $x \vee y \in K(L)$. Further, if x and y are comparable, for example, if $x \leq y$, then $x \wedge y = x$, $x \vee y = y \in K(L)$. □

Definition 2. *Let* $K(L)$ *be nonempty. If we denote* $K_0(L) = L$, $K_1(L) = K(L)$, *then we define recursively:*

- *the n-th nucleus of L:* $K_n(L) = K(K_{n-1}(L))$, $n \in N$,
- *the n-th coat of L:* $S_n(L) = K_{n-1}(L) \setminus K_n(L), n \in N$.

Definition 3. *A lattice* L *is:*

- *finitely reducible iff* $K_n(L) = \emptyset$ *for some* $n \in N$
- *a* K*-lattice iff* $K(L) = L$.

Of course, this classification does not include all the lattices.

Corollary 2. *A finite lattice is finitely reducible iff it does not contain a* K*-sublattice.*

Proof. The proof is direct. □

2. On Nucleus, Coats and Sublattices

One characterization of the nucleus of a lattice is given by:

Theorem 3. *Let* $K(L)$ *be the nucleus of the lattice* L. *Then it holds:*

(i) for every subset $B \subset L$, $K(L) \cup B$ *is a sublattice of* L;

(ii) $K(L)$ *is the minimal sublattice of* L *which satisfies (i).*

Proof. (i) Let $x, y \in K(L) \cup B$. Then, if x and y are comparable, for example if $x \leq y$, we have $x \wedge y = x$ and $x \vee y = y \in K(L) \cup B$. If x and y are incomparable, then $x \wedge y$, $x \vee y \in K(L) \cup B$, (Definition 1 gives $x \wedge y$, $x \vee y \in K(L)$).

(ii) Let H be a sublattice which satisfies:

$$\forall B \subset L,\ H \cup B \text{ is a sublattice of } L.$$

We have to show that $K(L) \subset H$. Let $x \in K(L)$. Then $x = y \wedge z$ or $x = y \vee z$ for some $y, z \in L$, where $y \| z$. If $x = y \wedge z$, we observe a set $H \cup \{z, y\}$, which is a sublattice by the hypothesis and $x = y \wedge z \in H \cup \{y, z\}$. Since $x \neq y$ and $x \neq z$, we have $x \in H$. Similarly, if $x = y \vee z$. So, $K(L) \subset H$. □

Lattices which are not K-lattices have a nonempty first coat $S_1(L)$. Now, we shall consider a relation between the first coat and a lattice of all the sublattices of a lattice L, Sub(L). For the lattice Sub(L) and the Boolean lattice $\mathcal{P}(S_1(L))$, we have:

Theorem 4. *Let L be a lattice and $S_1(L) \neq \emptyset$. Then the Boolean lattice $\mathcal{P}(S_1(L))$ is a homomorphic image of Sub(L).*

Proof. Let us define a map f: Sub(L) $\to \mathcal{P}(S_1(L))$ in the following way:

$$f(H) = H \cap S_1(L), \quad \text{for all } H \in \mathrm{Sub}(L).$$

(i) f is a homomorphism:
Let $H, K \in$ Sub(L). Then we have $f(H \wedge K) = f(H \cap K) = (H \cap K) \cap S_1(L) = (H \cap S_1(L)) \cap (K \cap S_1(L)) = f(H) \cap f(K)$. Further we have to prove that $f(H \vee K) = f(H) \vee f(K)$. Since $f(H \vee K) = f([H \cup K]) = [H \cup K] \cap S_1(L)$, and $f(H) \vee f(K) = f(H) \cup f(K) = (H \cap S_1(L)) \cup (K \cap S_1(L)) = (H \cup K) \cap S_1(L)$, we have to show that:

$$[H \cup K] \cap S_1(L) = (H \cup K) \cap S_1(L)$$

($\supset$) $H \cup K \subset [H \cup K]$ implies the first inclusion.
($\subset$) Let $x \in [H \cup K] \cap S_1(L)$. Then $x \in [H \cup K]$ and $x \in S_1(L)$. Suppose that $x \notin H \cup K$ and $x \in [H \cup K] \cap S_1(L)$. Since $x \in S_1(L)$, for a pair of incomparable elements $y, z \in L$ we have $x \neq y \wedge z$ and $x \neq y \vee z$. Now we shall observe a set $N = [H \cup K] \backslash \{x\}$. Let $a, b \in N$. If a and b are

comparable, then $a \wedge b$, $a \vee b \in N$. If they are incomparable, then $a \wedge b$ and $a \vee b$ are different from x and again $a \wedge b$, $a \vee b \in N$, so $H, K \subset N$ and N is a sublattice. But $[H \cup K]$ is the smallest sublattice containing H and K. Contradiction. So $x \in (H \cup K) \cap S_1(L)$.

(ii) f is onto:
Let $X \in \mathcal{P}(S_1(L))$. By Theorem 1 we have $H = K(L) \cup X \in \mathrm{Sub}(L)$. Also $f(H) = f(K(L) \cup X) = (K(L) \cup X) \cap S_1(L) = (K(L) \cap S_1(L)) \cup (X \cap S_1(L)) = \emptyset \cup X = X$. □

Let us denote the principal filter generated by $K(L)$ in the lattice $\mathrm{Sub}(L)$ by $[K(L))_{\mathrm{Sub}(L)}$, or briefly $[K(L))$. Then we have:

Theorem 5. *Let L be a lattice and $K(L) \neq \emptyset$. Then it holds:*

$$[K(L)) \cong \mathcal{P}(S_1(L)).$$

Proof. We shall prove that the map $f : [K(L)) \to \mathcal{P}(S_1(L))$ defined by:

$$f(H) = H \cap S_1(L),$$

is an isomorphism.

(i) f is a homomorphism:
If $H, K \in [K(L)$, we have $f(H \wedge K) = f(H \cap K) = (H \cap K) \cap S_1(L) = (H \cap S_1(L)) \cap (K \cap S_1(L)) = f(H) \cap f(K)$. Also, by Theorem 1, $H \cup K$ is a sublattice of L, so $[H \cup K] = H \cup K$. Therefore, $f(H \vee K) = f(H \cup K) = (H \cup K) \cap S_1(L) = (H \cap S_1(L)) \cup (K \cap S_1(L)) = f(H) \cup f(K)$.

(ii) f is "1 - 1":
Suppose that $f(H) = f(K)$. Then $H \cap S_1(L) = K \cap S_1(L)$. Now, we have $H = H \cap L = H \cap (K(L) \cup S_1(L)) = (H \cap K(L)) \cup (H \cap S_1(L)) = K(L) \cup (H \cap S_1(L))$. Similarly, we have $K = K(L) \cup (K \cap S_1(L))$, and $H \cap S_1(L) = K \cap S_1(L)$ implies $H = K$. Therefore f is "1 - 1".

(iii) f is onto:
Let $X \in \mathcal{P}(S_1(L))$. For $H = K(L) \cup X$ we have $f(H) = f(K(L) \cup X) = (K(L) \cup X) \cap S_1(L) = (K(L) \cap S_1(L)) \cup (X \cap S_1(L)) = \emptyset \cup X = X$. Therefore, f is onto. □

3. Nucleus, Coats and Automorphisms

For the proof of the following theorems we need the next known lemma:

Lemma 6 *Let L be a lattice and $f : L \to L$ an automorphism of L. Then:*

(i) for every $x, y \in L$, $x \| y$ iff $f(x) \| f(y)$.

(ii) if H is a sublattice of L and $f(H) = H$, then $f \mid H$ is an automorphism on H. □

Theorem 7. *Let L be a lattice and $f : L \to L$ an automorphism of L. Then:*

(i) $f(K(L)) = K(L)$,

(ii) if $S_1(L) \neq \emptyset$, then $f(S_1(L)) = S_1(L)$.

Proof. (i) Let $y \in f(K(L))$. Then there exists $x \in K(L)$, such that $y = f(x)$. Since $x \in K(L)$, we have for some incomparable $y', z' \in L$ $x = y' \wedge z'$ or $x = y' \vee z'$. If $x = y' \wedge z'$, then $y = f(x) = f(y' \wedge z') = f(y') \wedge f(z')$. By Lemma 6, $y' \| z'$ implies that $f(y') \| f(z')$ and $y \in K(L)$. Similarly for $x = y' \vee z'$. So, we have:

$$f(K(L)) \subset K(L).$$

($\supset$) Let $y \in K(L)$, and let us assume that $y = u \wedge w$, $u \| w$. f is onto and we have $u = f(u_1)$ and $w = f(w_1)$ for some u_1, $w_1 \in L$. Also, by Lemma 6, we have $u_1 \| w_1$. Now, $y = u \wedge w = f(u_1) \wedge f(w_1) = f(u_1 \wedge w_1)$. However, $u_1 \wedge w_1 \in K(L)$, becouse $u_1 \| w_1$ and we have $y \in f(K(L))$. So, it holds:

$$K(L) \subset f(K(L)).$$

(ii) Since f is "1 - 1" and $K(L) \cap S_1(L) = \emptyset$, we have $f(K(L)) \cap f(S_1(L)) = \emptyset$, that is $K(L) \cap f(S_1(L)) = \emptyset$. Since $L = K(L) \cup S_1(L)$ we conclude that $f(S_1(L)) \in S_1(L)$. f is onto, so $S_1(L) \subset f(S_1(L))$. □

Corollary 8. *Let L be a lattice and $S_1(L)$, $S_2(L) \ldots$ the nonempty coats of L. Then*

$$f(S_i(L)) = S_i(L), \quad \textit{for each } i = 1, 2, \ldots$$

Proof. $f(S_1) = S_1$ is proved. By Lemma 6 we have that $f \mid K(L)$ is an automorphism. Since $S_2(L) = S_1(K(L))$, from Theorem 7 it follows that $f(S_2(L)) = S_2(L)$ etc. □

Corollary 9. *The last nonempty coat of a finitely reducible lattice is a chain.*

Proof. Let L be a finitely reducible lattice. Then for some $n \in N, K_{n-1}(L) \neq \emptyset$, and $K_n(L) = \emptyset$. Since $S_n(L) = K_{n-1}(L) \setminus K_n(L) = K_{n-1}(L)$ is the last nonempty coat of L, and $K(S_n(L)) = K_n(L) = \emptyset$, in $S_n(L)$ there are no incomparable elements, so $S_n(L)$ is a chain. □

Corollary 10. *The last nonempty coat of a finite, and finitely reducible lattice L, is a subset of the set of all the fixed points for every automorphism of L.*

Proof. By Corollary 9, $S_n(L)$ is a finite chain and by Corollary 8, $f(S_n(L)) = S_n(L)$. The only automorphism of a finite chain is the identity map, so $f(x) = x$ for all $x \in S_n(L)$. □

Theorem 11. *Let L_1 and L_2 be lattices and $f : L_1 \to L_2$ an onto homomorphism. Then*

$$K(L_2) \subset f(K(L_1)).$$

Proof. Let $y \in K(L_2)$. Then there exist $u, w \in L_2$, $u \| w$, such that $y = u \wedge w$ or $y = u \vee w$. Assume $y = u \vee w$. Since f is onto, there exist $u', w' \in L_1$ such that $f(u') = u$ and $f(w') = w$. Suppose that u' and w' are comparable, for example $u' \leq w'$. Then $u' \wedge w' = u'$ and $f(u' \wedge w') = f(u') \wedge f(w') = f(u')$, that is $f(u') \leq f(w')$, so $u \leq w$. Contradiction. So $u' \| w'$ and we have $u' \vee w' = x \in K(L_1)$. Now we have $y = u \vee w = f(u') \vee f(w') = f(u' \vee w') = f(x)$, where $x \in K(L_1)$. So $y \in f(K(L_1))$. □

4. Nucleus and Direct Products

Theorem 12. *For unbounded lattices L_1 and L_2, $L_1 \times L_2$ is a K-lattice.*

Proof. Let $(a, x) \in L_1 \times L_2$. Since L_1 and L_2 are unbounded, there exist $b \in L_1$ and $y \in L_2$ such that $a < b$, and $x < y$. Then we have $(a, y) \wedge (b, x) = (a \wedge b, y \wedge x) = (a, x)$. However (a, y) and (b, x) are incomparable, and we have $(a, x) \in K(L_1 \times L_2)$ □.

Theorem 13. *Let L_1 and L_2 be lattices and $|L_1|, |L_2| > 1$. Then:*

$$L_1 \times L_2 \backslash \{(0,1),(1,0)\} \subset K(L_1 \times L_2).$$

Proof. Let $x = (a,x) \in L_1 \times L_2 \backslash \{(0,1),(1,0)\}$. Then it is possible:

1. $a = 0$
   Then $x < 1$ and $(a,x) = (0,1) \wedge (1,x)$. Since $0 < 1$ and $1 > x$, the elements $(0,1)$ and $(1,x)$ are incomparable, so $(a,x) \in K(L_1 \times L_2)$.

2. $0 < a < 1$
   Then it is possible:
   2.1. $x = 0$
   Then $(a,x) = (a,1) \wedge (1,0)$, $(a,1) \| (1,0)$. So, $(a,x) \in K(L_1 \times L_2)$.
   2.2. $0 < x < 1$
   Then $(a,x) = (1,x) \wedge (a,1)$, $(1,x) \| (a,1)$. So, $(a,x) \in K(L_1 \times L_2)$.
   2.3. $x = 1$
   Then $(a,x) = (a,0) \vee (0,1)$, $(a,0) \| (a,1)$. So, $(a,x) \in K(L_1 \times L_2)$.
3. $a = 1$.

Then $(a,x) = (1,0) \vee (0,x)$, $(1,0) \| (0,x)$. So, $(a,x) \in K(L_1 \times L_2)$. □

Theorem 14. *Let L_1 and L_2 be lattices with 0 and 1. Then it holds:*

(i) $(0,1) \in K(L_1 \times L_2)$ *iff 0 is $\wedge$-reducible or 1 is $\vee$-reduciable*

(ii) $(1,0) \in K(L_1 \times L_2)$ *iff 1 is $\vee$-reducible or 0 is $\wedge$-reducible.*

Proof. (i) ($\Longrightarrow$) Let $(0,1) \in K(L_1 \times L_2)$. Suppose that $(0,1) = (a,x) \vee (b,y)$ and $(a,x) \| (b,y)$. Then $a \vee b = 0$ implies $a = 0$ and $b = 0$. Also we have $x \vee y = 1$. If x and y are comparable, for example $x \leq y$, then $(a,x) \leq (b,y)$. Contradiction. So $x \| y$ and $x \vee y = 1$. 1 is $\vee$-reducible. Similarly for $(0,1) = (a,x) \wedge (b,y)$ we have that $x = y = 1$ and $0 = a \wedge b$, where $a \| b$, hence 0 is $\wedge$-reducible.

($\Longleftarrow$) If $0 = a \wedge b, a \| b$, then $(0,1) = (a,1) \wedge (b,1)$ and $(a,1) \| (b,1)$. Therefore $(0,1)$ is $\wedge$-reducible and $(0,1) \in K(L_1 \times L_2)$. Similarly if 1 is $\vee$-reducible.

(ii) Analogously. □

Corollary 15. *The direct product of K-lattices is a K-lattice.*

Proof. Since every element of a K-lattice is $\wedge$-reducible or $\vee$-reducible, such are 1 and 0, if they exist, and from Theorems 14 and 13 follows the proposition. □

Corollary 16. *Let L be a lattice. Then $L \times B_4$ is a K-lattice, where by B_4 we denote the four element Boolean lattice.*

Proof. In the lattice B_4, 0 is $\wedge$-reducibile and 1 is $\vee$-reducibile. So, by Theorem 14, $(0,1),(1,0) \in K(L \times B_4)$, and by Theorem 13, $K(L \times B_4) = L \times B_4$. □

5. On the Characterization of K-lattices

Theorem 17. *Every lattice can be embedded in some K-lattice.*

Proof. Let L a lattice. By Corollary 16, $L_1 = L \times B_4$ is a K-lattice. If $L' = L \times \{0\} \subset L_1$, then for $(x,0),(y,0) \in L'$ we have $(x,0) \wedge (y,0) = (x \wedge y, 0)$ and $(x,0) \vee (y,0) = (x \vee y, 0) \in L'$, so L' is a sublattice of L_1. Also, the mapping $f : L \to L'$ given by $f(x) = (x,0)$ is an isomorphism. Now L' is a sublattice of L_1 isomorphic to L. □

If we denote by $\mathcal{K}$ the class of all the K-lattices, then we have

Theorem 18. *$\mathcal{K}$ is not an equational class.*

Proof. Let $L \in \mathcal{K}$. Then $|L| > 1$ and there exist at least two comparable elements of L, i.e. there exist $a,b \in L$ such that $a < b$. For $H = \{a,b\}$ we have that H is a sublattice of L and $H \notin \mathcal{K}$.

Theorem 19. *There is no equational class $\mathcal{K}_1$ different from the class of all the lattices, such that $\mathcal{K} \subset \mathcal{K}_1$.*

Proof. Similar to the proof of the previous theorem. □

Theorem 20. *$L \in \mathcal{K}$ iff for all $x \in L$, $L \backslash \{x\}$ is not a sublattice of L.*

Proof. ($\Longrightarrow$) Let $L \in \mathcal{K}$ and $x \in L$. Then $x \in K(L)$ and there exist $y, z \in L$ such that $y \| z$ and $x = y \wedge z$ or $x = y \vee z$. Let $x = y \wedge z$. Now we have $y, z \in L \backslash \{x\}$ and $y \wedge z \notin L \backslash \{x\}$, so $L \backslash \{x\}$ is not a sublattice of L.

($\Longleftarrow$) Suppose that the condition is satisfied and that $L \notin \mathcal{K}$. Then $K(L) \neq L$, i.e. $S_1(L) \neq \emptyset$. If $x \in S_1(L)$, by Theorem 3 (i), we have that $K(L) \cup S_1(L) \backslash \{x\} = L \backslash \{x\}$ is a sublattice of L. Contradiction. □

Corollary 17. *Let L be a finite lattice. Then $L \in \mathcal{K}$ iff L does not contain a sublattice H of cardinality $|L| - 1$.* □

References

[1] Grätzer, G.: Lattice Theory: First Concepts and Distributive Lattices. San Francisco Cal.: Freeman 1971.

REZIME

JEZGRO MREŽE

Posmatran je skup razloživih elemenata proizvoljne mreže L, u oznaci $K(L)$, i veza sa podmrežama, morfizmima i proizvodima. Klasa mreža za koje je $K(L) = L$ nije jednakosna klasa. Svaka mreža može se potopiti u neku mrežu iz ove klase.

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