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SOME INEQUALITIES FOR SPECTRAL RADIUS IN CONNECTION WITH RELAXATION METHODS CONVERGENCE THEORY

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Abstract

A very important thing in the convergence theory of relaxation methods is to obtain some upper bounds for spectral radius of the iterative matrix. In this paper two upper bounds for the spectral radius of the iteration matrix of the USSOR (unsymmetric succesive overrelaxation) method are derived.

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1. Introduction

USSOR (unsymmetric succesive overrelaxation) method for solving a linear system

$$Ax = b,$$

where $A \in \mathbf{R}^{n,n}$ is a regular matrix with the property $a_{ii} = 1$, $i = 1, 2, \dots, n$, $b \in \mathbf{R}^n$, has the following form

$$x^0 \in \mathbf{R}^n; \; x^{k+1} = S_{\sigma\omega} x^k + d, \; k = 0, 1, \dots,$$

where

$$S_{\sigma\omega} = (E-\sigma L)^{-1}((1-\sigma)E+\sigma U)(E-\omega U)^{-1}((1-\omega)E+\omega L),$$

$$d = (E-\omega U)^{-1}((1-\omega)E+\omega L)(E-\sigma L)^{-1}\sigma b + (E-\omega U)^{-1}\omega b,$$

σ, ω are nonzero real parameters and $A = E - L - U$ is the standard splitting of the matrix A into its diagonal (E), strictly lower (L) and strictly upper (U) triangular parts.

The usefullness of this method is described in [7], see also [5] , [1] , [6] , [3] .

Convergence area for USSOR method obtained in [3] is the same for all matrices from the set $\Omega(A) = \{B \in \mathbf{C}^{n,n} \mid |b_{ij}| = |a_{ij}|,\ i,j = 1,2,...,n\}$.

But, this convergence area can be improved for some special subclasses of $H-$ matrices. In order to do this we have to obtain some upper bounds for spectral radius $\rho(S_{\sigma\omega})$. If we prove $\rho(S_{\sigma\omega}) \leq \epsilon$, then by analysing the condition $\epsilon < 1$, we shall obtain sufficient conditions for convergence.

In this paper we shall prove two different upper bounds for spectral radius of the iteration matrix $S_{\sigma\omega}$.

2. Upper bounds for spectral radius

The iteration matrix of the USSOR method has the following form

$$S_{\sigma\omega} = (E-\sigma L)^{-1}((1-\sigma)E+\sigma U)(E-\omega U)^{-1}((1-\omega)E+\omega L).$$

From now on we shall use the following notations:

$$P_i(A) = \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}| \quad i = 1,2,...,n, \quad \text{where} \quad A = [a_{ij}] \in \mathbf{R}^{n,n};$$

$$\ell_i = P_i(L), \quad i = 1,2,...,n; \quad u_i = P_i(U), \quad i = 1,2,...,n;$$

$$\epsilon(t,x,y) = \frac{|1-t| + |t|\,x}{1 - |t|\,y} \quad \text{for} \quad t,x,y \in \mathbf{R}.$$

Theorem 1. *If* $1 - |\sigma|\,\ell_i > 0$, $1 - |\omega|\,u_i > 0$, $i = 1,2,...,n$, *then*

$$\rho(S_{\sigma\omega}) \leq max_i\epsilon(\sigma, u_i, \ell_i) \cdot max_i\epsilon(\omega, \ell_i, u_i).$$

Proof. Since

$$\rho(S_{\sigma\omega}) \leq \|S_{\sigma\omega}\|_\infty \leq \|S_\sigma^1\|_\infty \|S_\omega^2\|_\infty,$$

where

$$S_\sigma^1 = (E - \sigma L)^{-1}((1-\sigma)E + \sigma U),$$
$$S_\omega^2 = (E - \omega U)^{-1}((1-\omega)E + \omega L),$$

$\|\cdot\|_\infty$ is the maximum norm, it is sufficient to show that

$$\|S_\sigma^1\|_\infty \leq max_i \epsilon(\sigma, u_i, \ell_i)$$

and

$$\|S_\omega^2\|_\infty \leq max_i \epsilon(\omega, \ell_i, u_i).$$

The first inequality can be treated as a special case ($\sigma = \omega$) of the statement from [2]. This upper bound is obtained by using Sassenfeld's criteria. The same technique can be used for obtaining the second inequality and we shall present it here:

Let y be a vector for which is

$$\|S_\omega^2\|_\infty = \|S_\omega^2 y\|_\infty; \quad \|y\|_\infty = 1.$$

Let $z = S_\omega^2 y$. Then for each $i = 1, 2, ..., n$ holds:

$$(((1-\omega)E + \omega L)y)_i = ((E - \omega U)z)_i, \quad \text{i.e.}$$

$$z_i + (\omega - 1)y_i = \omega(\sum_{j=1}^{i-1} a_{ij}y_j + \sum_{j=i+1}^{n} a_{ij}z_j).$$

If we denote

$$p_n = \ell_n,$$

$$p_i = \sum_{j=1}^{i-1} | a_{ij} | + \sum_{j=i+1}^{n} (| 1-\omega | + | \omega | p_j) | a_{ij} |, \quad i = n-1, n-2, ..., 2, 1,$$

by using mathematical induction we shall prove

(1) $\quad | z_i + (\omega-1)y_i | \leq | \omega | p_i$ and $| z_i | \leq | 1-\omega | + | \omega | p_i$, $i = n, n-1, ..., 2, 1.$

For $i = n$

$$| z_n + (\omega - 1)y_n | = | \omega | \, | \sum_{j=1}^{n-1} a_{nj}y_j | \leq | \omega | \ell_n = | \omega | p_n,$$

because of $|y_i| \leq 1, i = 1, 2, ..., n$.

Now,

$$|z_n| - |(\omega - 1)y_n| \leq |z_n + (\omega - 1)y_n| \leq |\omega| p_n, \quad \text{and}$$

$$|z_n| \leq |1 - \omega| + |\omega| p_n.$$

Suppose that (1) is true for all $i = n, n-1, ..., k+1$, and prove that it is true for $i = k$.

$$|z_k + (\omega - 1)y_k| \leq |\omega| (\sum_{j=1}^{k-1} |a_{kj}| + \sum_{j=k+1}^{n} |a_{kj}| (|1 - \omega) + |\omega| p_j)) = |\omega| p_k.$$

Obviously, then is

$$|z_k| \leq |1 - \omega| + |\omega| p_k.$$

Proof of (1) is now completed. □

If we use a different technique, we can prove better upper bound for spectral radius $\rho(\mathcal{S}_{\sigma\omega})$.

From now on we shall denote

$$\alpha_{ii} = (UL)_{ii}, \ z_i = P_i(UL) + |\alpha_{ii}|, \ i = 1, \ldots, n.$$

Theorem 2. *If* $1 - |\omega| u_i - |\sigma| l_i - |\omega\sigma| z_i > 0, \ i = 1, \ldots, n$, *then*

$$\rho(\mathcal{S}_{\sigma\omega}) \leq \max_{1 \leq i \leq n} \frac{|1-\sigma||1-\omega| + |\omega||1-\sigma| l_i + |\sigma||1-\omega| u_i + |\sigma\omega| z_i}{1 - |\omega| u_i - |\sigma| l_i - |\omega\sigma| z_i}.$$

Proof. Suppose that there existsan eigenvalue λ of the matrix $\mathcal{S}_{\sigma\omega}$ such taȟt for each $i = 1, \ldots, n$ the inequality

$$|\lambda| > \frac{|1-\sigma||1-\omega| + |\omega||1-\sigma| l_i + |\sigma||1-\omega| u_i + |\sigma\omega| z_i}{1 - |\omega| u_i - |\sigma| l_i - |\omega\sigma| z_i}$$

is true.

Then

$$|\lambda - (1-\sigma)(1-\omega) + (\lambda - 1)\sigma\omega\alpha_{ii}| \geq |\lambda| - |1-\sigma||1-\omega| - (|\lambda| + 1)|\sigma\omega||\alpha_{ii}| >$$

$$> (|\omega||1-\sigma| + |\lambda\sigma|) l_i + (|\sigma||1-\omega| + |\lambda\omega|) u_i + (|\lambda| + 1)|\sigma\omega|(z_i - |\alpha_{ii}|) \geq$$

$$\geq |\omega(1-\sigma)+\lambda\sigma|l_i + |\sigma(1-\omega)+\lambda\omega|u_i + |\lambda-1||\sigma\omega|(z_i - |\alpha_{ii}|),$$

for each $i = 1, \dots, n$.

This inequality can be written as

$$|(C)_{ii}| > P_i(C), \; i = 1, \dots, n,$$

where

$$C = (\lambda-(1-\sigma)(1-\omega))E-(\lambda\omega+\sigma(1-\omega))U-(\lambda\sigma+\omega(1-\sigma))L+(\lambda-1)\sigma\omega UL.$$

So, the matrix C is an SDD matrix and $det(C) \neq 0$.

Because of

$$((1-\sigma)E+\sigma U)(E-\omega U)^{-1} = (E-\omega U)^{-1}((1-\sigma)E+\sigma U),$$

we have

$$\mathcal{S}_{\sigma\omega} = (E-\sigma L)^{-1}(E-\omega U)^{-1}((1-\sigma)E+\sigma U)((1-\omega)E+\omega L) =$$

$$= (E-\omega U-\sigma L+\omega\sigma UL)^{-1}((1-\sigma)(1-\omega)E+\omega(1-\sigma)L+\sigma(1-\omega)U+\sigma\omega UL)$$

and

$$det(\lambda E - \mathcal{S}_{\sigma\omega} = det(E-\omega U-\sigma L+\omega\sigma UL)^{-1}det(C).$$

Hence, if λ is an eigenvalue of the matrix $\mathcal{S}_{\sigma\omega}$, then $det(C) = 0$. This contradicts to the previous conclusion that matrix C is regular. □

3. Numerical Examples

The inequalities obtained in Theorem 1 and Teorem 2 can be used for convergence analysis of the USSOR method.

Here we will not investigate special subclasses of H-matrices and their convergence areas. We present only the numerical example where it is shown that σ and ω can be choosen such that upper bounds from Theorem 1 and Theorem 2 are both less than 1, while these parameters do not belong to te convergence area obtained in Theorem 5 from [3] .

Example. Let

$$A = \begin{bmatrix} 0 & 1/4 \\ 1/9 & 0 \end{bmatrix}.$$

For this matrix $\rho(|L| + |U|) = \frac{1}{6}$ and the convergence area obtained in Theorem 5 from [3] has te following form

$$\sigma \in (-2.5, 3.5),\ \omega \in \mathcal{O}_\omega.$$

Obviously, the pair $(\sigma, \omega) = (4, 1)$ does not belong to this area.

But, if we put $\sigma = 4$ and $\omega = 1$ into the upper bound for $\rho(\mathcal{S}_{\sigma\omega})$ given in Theorem 1, we obtain

$$\rho(\mathcal{S}_{\sigma\omega}) = \rho(\mathcal{S}_{41}) \leq max_{i=1,2}\frac{3+4u_i}{1-4l_i} \cdot max_{i=1,2}\frac{l_i}{1-u_i} =$$

$$= max\{4, \frac{27}{5}\} \cdot max\{0, \frac{1}{9}\} = \frac{27}{45} < 1.$$

Also, from Theorem 2 we obtain

$$\rho(\mathcal{S}_{\sigma\omega}) = \rho(\mathcal{S}_{41}) \leq max_{i=1,2}\frac{3l_i + 4z_i}{1-4l_i-u_i-4z_i} =$$

$$= max\{\frac{4}{23}, \frac{3}{5}\} = \frac{3}{5} < 1.$$

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REZIME

NEKE NEJEDNAKOSTI ZA SPEKTRALNI RADIJUS U VEZI SA TEORIJOM KONVERGENCIJE RELAKSACIONIH POSTUPAKA

Za teoriju konvergencije relaksacionih postupaka veoma je važno odrediti neka gornja ograničenja za spektralni radijus iterativne matrice. U ovom radu dokazane su dve gornje granice za spektralni radijus iterativne matrice USSOR postupka.

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