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THE LAGUERRE EXPANSION IN S'_+ AND THE MELLIN TRANSFORM

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Abstract

In the first part of the paper a space of generalized functions whose have elements orthonormal expansions into Laguerre series is studied. This space is in fact the space S'_+ and by using the orthogonormal expansions, in the second part of the paper the modified Mellin transform on S'_+ is defined and investigated. Two inversion formulas for this transform are also given.

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1. Introduction

Monograph [10] by A. H. Zemanian is of essential importance for the theory of generalized function integral transforms. For a number of classical integral transforms Zemanian defined the appropriate basic spaces which contain kernels of these transforms and in this way he defined, by the so-called dierect method, generalized integral transforms on dual spaces. The basic spaces appropriate for integral transforms often have not some other properties which are important in the theory of generalized functions and their applications. For example, basic spaces are often not closed with respect to the operation of ordinary differentiation. Because of that, investigations of generalized integral transforms on the well-known subspaces of the space of Schwartz's distributions are important, as well. The space of tempered distributions with their supports contained in $[0,\infty)$, S'_+, has many good properties. This is a convolution algebra; Fourier, Laplace, Stieltjes and Hilbert transforms have useful properties on this space. From the point of view of the series expansions we shall investigate in this paper the modified Mellin transform in S'_+. In Section 1. we shall recall from [10] and [9] the basic facts concerning the A'-type spaces whose elements have ortho-normal Laguerre series expansions, which is, in fact, the space S'_+ ([6]). Moreover, we shall give, in a somewhat different way than in [10], the structural properties of the basic space and its dual. In the next two sections we introduce and investigate the modified Mellin transform on S'_+, M-transform. This transform mapps S'_+, into the space of the Newton series. We shall give several properties and two inversion formulas for the M-transform which make the development of the operational calculus possible. The first inversion formula is of practical use and of interest from the numerical point of view and the second one is a typical distributional result.

2. Laguerre expansion of elements from S'_+

It is well-known that the space of tempered distributions S'_+ on the real line $\mathbf{R}$ is an A'-type space. Recall, A'-type spaces were introduced and studied in [10. Ch. 9]. The A'-type spaces whose elements hace unique orthonormal expansions into the Laguerre series were studied by Zayed [9], who showed that this space is exactly the space S'_+.

We shall recall some basic facts conserning this space from [9] and [10], and redefine them in a somewhat different way.

We denote by $l_n, n \in \mathbf{N}_0, (\mathbf{N}_0 = \mathbf{N} \cup \{0\})$ the Laguerre orthonormal base of the space $L^2(\mathbf{R}_+)(\mathbf{R}_+ = (0,\infty), \overline{\mathbf{R}}_+ = [0,\infty))$. These functions are defined on $\mathbf{R}_+$ by $l_n(t) = e^{-t/2}L_n(t)$, where $L_n(t) = \sum_{m=0}^{n} (\begin{pmatrix} n \\ n-m \end{pmatrix} \frac{(-t)^m}{m!}, n \in \mathbf{N}_0$.

The testing functions space LG is defined as the space of all the smooth functions ϕ defined on $\mathbf{R}_+(\phi \in C^\infty(\mathbf{R}_+))$, such that $\|R^k\phi\|_0 = (\int_0^\infty |R^k \phi(t)|^2 dt)^{1/2} < \infty$, $k \in \mathbf{N}_0$, and $\langle R^k\phi, l_n\rangle = \langle\phi, R^k l_n\rangle = (-n)^k\langle\phi, l_n\rangle$, $k \in$

$\mathbf{N}_0, n \in \mathbf{N}_0$, where $R = e^{t/2}\ Dte^{-t}\ De^{t/2}\ (D = d/dt), R^{k+1} = R(R^k), k \in \mathbf{N}_0, R^0$ - the identity operator, and $\langle \phi, \psi$

rangle $= (\ \phi, \overline{\psi}) = \int_0^\infty \phi(t)\psi(t)dt, \phi, \psi \in L^2(\mathbf{R}_+)$.

The space LG is, in fact, the space of all the $\phi \in C^\infty(\overline{\mathbf{R}}_+)$ for which the norms

$$\sup\{t^k|\phi^{(j)}(t)|; t \in [0, \infty), j \leq k\},\ k \in \mathbf{N}_0,$$

are all finite. This follows from [9] and [6]. In the notation from [5, p. 24], we have $LG = S(\overline{\mathbf{R}}_+)$ and $(LG)' = S'(\overline{\mathbf{R}}_+)$, where the last space is the space of all the tempered distributions with supports in $[0, \infty)$. We shall use the more usual notation for this space, S'_+.

Let $S_k,\ k \in \mathbf{R}$ be the space of all the formal series

$$\phi = \sum_{n=0}^{\infty} a_n l_n \text{ such that } \|\phi\|_k = (|a_0|^2 + \sum_{n=1}^{\infty} |a_n|^2 n^{2k})^{1/2} < \infty.$$

Clearly, for $k \geq 0, S_k \subset L^2$.

From [10, Ch. 9], there follows $S(\overline{\mathbf{R}}_+) = \text{proj.} \lim_{k\to\infty} S_k$.

Proposition 1.

i. Spaces $S_k,\ k \in \mathbf{R}$ are B-spaces.

ii. Inclusion mappings $i_{k,l} : S_k \to S_l, k > l$, are compact.

iii. The set S of polynomials of the form $\sum_{m=0}^{n} c_m l_m$, (c_m are complex numbers), is dense in S_k for every $k \in \mathbf{R}$.

Proof. (i) (iii) are obvious and (ii) follows from the Kolmogorof Theorem [3, p. 79].

Proposition 2. $\overset{\circ}{S}{}' (\overline{\mathbf{R}}_+) =$ *ind.* $\lim_{k\to\infty} S_k$ *in the sense of strong topologies.*

Proof. This follows from [3, p. 146]

One can easily show that the dual of $S_k,\ k \in \mathbf{R}$, can be identified with the space S_{-k}. A continuous linear functional on S_k, f is identified with the series $\sum_{n=0}^{\infty} b_n l_n$ whose coefficients are determined by $b_n = \langle f, l_n\rangle, n \in \mathbf{N}_0$. The duality between S_{-k} and S_k is given by

$$(f,\phi) = \langle \sum_{n=0}^{\infty} b_n l_n, \sum_{n=0}^{\infty} \bar{a}_n l_n\rangle = \sum_{n=0}^{\infty} b_n \bar{a}_n.$$

Obviously $\sum_{n=0}^{\infty} b_n l_n$ converges to f in S'_+ as $m \to \infty$.

Let $f_n = \sum_{m=0}^{\infty} b_m^n l_m$, $n \in \mathbf{N}$, and $f = \sum_{m=0}^{\infty} b_m l_m$ be from S'_+. It follows form Proposition 1 that the weak and strong convergences of a sequence in $S'_+ = (S(\overline{\mathbf{R}}_+))'$ are equivalent and that $f_n \to f$, if there exists $k \in \mathbf{N}_0$ such that

$$|b_0^n - b_0| + \sum_{m=1}^{\infty} |b_m^n - b_m|^2 m^{2k} \to 0 \text{ as } n \to \infty.$$

Note, if we consider f_n and f as elements from $S' = (S(\overline{R})')$, then $f_n \to f$ in $S'(\overline{R}_+)$ if this sequence converges in S'. If we consider R^k, $k \in \mathbf{N}_0$, as a mapping from S to $L^2(\mathbf{R}_+)$, we see that this mapping can be extended uniquely to the whole S_k to be linear and continuous by

$$\overline{R}^k \phi = \lim_{m\to\infty} R^k(\sum_{n=0}^{m} a_n l_n) = \sum_{n=0}^{m} a_n(-n)^k l_n, \text{ where } \phi \stackrel{2}{=} \sum_{n=0}^{\infty} a_n l_n \in S_k.$$

If $\phi \in S(\overline{\mathbf{R}}_+)$, then $\overline{R}^k \phi = R^k \phi$. So we shall denote this extention again by R^k. Obviously, the mapping $R^k : S_k \to S_0 = L^2(\mathbf{R}_+)$ is continuous and if $\phi \stackrel{2}{=} \sum_{n=1}^{\infty} a_n l_n \in S_k$, then $\|R^k\phi\|_0 = \|\phi\|_k$.

3. Modified Mellin transform

We denote by $\varphi_s, s \in \mathbf{C}$, the function on $\mathbf{R}_+$ defined by $t \to \varphi_s(t) = t^{s-1}e^{-t/2}/\Gamma(s)$. (Note that $a/\Gamma(s), s \in \mathbf{C}$, is an entire function).

Let $s \in \mathbf{C}$ and Re$s > 0$. We have

$$a_n = \int_0^{\infty} \varphi_s(t) l_n(t) dt = \frac{(-1)^s(s-1)\dots(s-n)}{n!} = \tag{1}$$

$$(-1)^n \binom{s-1}{n}, n \in \mathbf{N}, a_0 = 1.$$

Since $(-1)^n \begin{pmatrix} s-1 \\ n \end{pmatrix} = \Gamma(n-s+1)/(n!\Gamma(1-s))$ and for a sufficiently large $n, |\Gamma(n-s+1)| \leq \Gamma(n-\mathrm{Re}s+1)^{n-\mathrm{Re}s+1-1/2}\exp(-n+\mathrm{Re}s-1)(1+0(1))$,

$$\left|\begin{pmatrix} s-1 \\ n \end{pmatrix}\right| \leq Cn^{-\mathrm{Re}s}, n > n_0.$$

This implies that for $\mathrm{Re}s > 1/2, \varphi_s \in L^2$ and $\varphi_s \stackrel{2}{=} \sum_{n=0}^{\infty} a_n l_n$, where a_n are determined by (1); moreover, $\varphi_s \in S_k, k \in \mathbf{R}$, if $\mathrm{Re}s > k + 1/2$. Let us put $D^k = \{s \in C | \mathrm{Re}s > k + \frac{1}{2}\}, k \in \mathbf{R}$.

Proposition 3. *The mapping $s \to \varphi_s$ is a holomorphic mapping from D_k into $S_k, k \in \mathrm{N}_0$.*

Proof. Let $k = 0$. By using Fybini's Theorem and Morera's Theorem one can prove that the mapping $D_0 \ni s \to \varphi_s \in S_0$ is weakly holomorphic, hence, strongly holomorphic.

Note that for any $s \in \mathbf{C}, R\varphi_s = (1-s)(\varphi_s - \varphi_{s-1})$.
This identity implies that φ_s is in the domain of R, i.e. in S_1, if the left hand side is in S_0. This means that s must belong to D_1. Since the mapping $D_1 \ni s \to \varphi_s \in S_1$ is the composition of the following mappings:

$$s \to s-1 \to (1-s)(\varphi_s - \varphi_{s-1}) = R\varphi_s \to \varphi_s,$$

we obtain that this mapping is holomorphic because the mapping $s \to s-1$ and $s-1 \to (1-s)(\varphi_s - \varphi_{s-1})$ are holomorphic and $R\varphi_s \to \varphi_s$ is continuous, where $\{R\varphi_s; s \in D_1\}$ and $\{\varphi_s; s \in D_1\}$ are observed as subspaces of S_0 and S_1 respectively. Namely, note that in the expansion of $\varphi_s, |a_1| > \frac{1}{2}$, and this implies $||\varphi_s||_0 \leq 5||R$

i $||_0, s \in D_1$. Now, the assertion follows by induction.

Proposition 4. *The mapping $s \to \varphi_s$ form D_k into $S_k, k \in \mathbf{R}$, is holomorphic.*

Proof. Let $r \in \mathbf{R}$ and $r_0 \in \mathrm{N}_0$ such that $r \leq r_0$. The mappings $s \to \varphi_s$ from D_r into S_r is the analytic continuation of the corresponding mapping form D_{r_0} into S_r, which is holomorphic by Proposition 3.

Let $f \in S'_+$. There is a $k_0 \in \mathbf{R}$ such that $f \in S'_k$ for $k \geq k_0$. Proposition 4 implies that $s \to \langle f(t), \varphi_s(t)\rangle, s \in D_{k_0}$, is holomorphic mapping.

The modofied Mellin transform of an $f \in S'_k, k \in \mathbf{R}$, is defined by this mapping, i.e.

$$s \to (Mf)(s) = \langle (t), \varphi_s(t)\rangle, s \in D_k. \tag{2}$$

Properties of this transform are given in

Proposition 5. *Let* $s \in D_k, k \in \mathbf{R}$ *and* $f = \sum_{n=0}^{\infty} b_n l_n \in S'_k$.

i.

$$(Mf)(s) = \sum_{n=0}^{\infty}(-1)^n \frac{(s-1)\dots(s-n)}{n!} b_n, \left(\left(\begin{array}{c} s-1 \\ 0 \end{array}\right) = 1\right),$$

where the series converges is an ordinary sense. Moreover, this series converges absolutely and uniformly on the mentioned domain. (Since this series converges for $s \in \mathbf{N}$, *we assume that* $(Mf)(m), m \in \mathbf{N}, m \leq k + 1/2$ *is defined by (i), as well).*

ii.

$$(MDf)(s+1) = -(Mf)(s) + \frac{1}{2}(Mf)(s+1), (M(tf))(s-1) =$$
$$= (s-1)(Mf)(s), (M(te^{t/2}(e^{-t/2}f)'))(s) =$$
$$= -s(Mf)(s), (M(e^{t/2}(e^{-t/2}f))(s) =$$
$$-(Mf)(s-1), (MRf)(s) = (1-s)((Mf)(s) - (Mf)(s-1)).$$

iii.

$$f(t) = \sum_{n=0}^{\infty}(-1)^n \Delta^n (Mf)(1) l_n(t)$$

in the sense of the convergence in S'_k, *where* Δ^n *is the n-th difference.*

Proof. (i) follows from (1) and the definition of Mf. (ii) is an easy calculation and (iii) follows form

$$(-1)^n b_n = \Delta^n (Mf)(1) = \sum_{m=0}^{\infty}(-1)^{n-m}\left(\begin{array}{c} n \\ m \end{array}\right)(Mf)(m+1), n \in \mathbf{N}_0([1]).$$

Let us remark that the formula given in Proposition 5 (iii) is an inversion formula for the M-transform which enables us to make operational calculus easier than with the generalised function inversion formula which will be given in the next proposition.

In [10, Ch.4] Zemanian defined the Mellin transform denoted here by M_0, on the dual of the corresponding basic space $M_{a,b}$ which contains φ_s for $s \in \{s \in \mathbf{C};\ \text{Re}s \in (a,b)\}, -\infty < a < b < \infty$. Obviously, $S(\overline{\mathbf{R}}_+) \subset M_{-1,b}$, but the converse inclusion does not hold. For any $a, b (a < b), S(\overline{\mathbf{R}}_+)$ does not contain $M_{a,b}$.

Note, if f is a locally integrable function on $\mathbf{R}_+$ and $t \to f(t)^{s-1}$ is from $L^1(\mathbf{R}_+)$ for $s \in \{s \in \mathbf{C};\ \text{Re}s\ \in (a,b)\}$ then

$$(M_0 f)(s) = \int_0^\infty f(t) t^{s-1} dt, a < \text{ Re}s\ < b \ [10, p.144].$$

Let $f \in S'_+$. It is well-known that for some continuous function of slow geowth F with supp $F \subset \overline{\mathbf{R}}_+$ and some $m \in \mathbf{N}_0, f$ is equal to $F^{(m)}$. For sufficiently large $r \in \mathbf{N}_0, F^{(\alpha)} \in S'_r, \alpha = 0, \ldots, m$, and $\varphi_{s-m} \in S_r$ if $s \in D_{r+m}$. This implies that $\varphi_{s-j} \in S_r$ for $j = 0, 1, \ldots, m-1$, if $s \in D_{r+m}$, as well. Thus, for $s \in D_{r+m}$ we have

$$(Mf)(s) = (-1)^m \langle F(t)(\varphi_s(t))^{(m)} \rangle = \frac{1}{2^m} \sum_{j=0}^{m} \binom{m}{j} (-2)^j \frac{(s-1)\ldots(s-j)}{\Gamma(s)}$$

$$\langle F(t), t^{s-j-1} e^{-t/2} \rangle = \frac{1}{2^m} \sum_{j=0}^{m} \binom{m}{j} (-2)^j (MF)(s-j).$$

The function $x \to F(x)e^{-x/2}, x \in \mathbf{R}$, is Mellin - transformable and for $s \in D_{r+m}$ we have

$$\Gamma(s-j)(Mf)(s-j) = M_0(Fe^{-x/2})(s-j),\ j = 0, \ldots, m.$$

This leads to the following formula

$$(3) \quad (Mf)(s) = 2^{-m} \sum_{j=0}^{m} \binom{m}{j} (-2)^j \frac{M_0(Fe^{-x/2})(s-j)}{\Gamma(s-j)},\ s \in D_{r+m}.$$

The formula enables us to obtain the inversion formula for the M-transform very similar to [10, Theorem 4.3] for M_0.

Proposition 6. *Let $f \in S'_+(\overline{\mathbf{R}}_+)$ and (Mf) be defined on $D_k, k \in \mathbf{R}$. There is $\sigma_0 > k + \frac{1}{2}$ such that for every $\sigma > \sigma_0$*

$$\langle \frac{e^{x/2}}{2\pi i} \int_{\sigma-ir}^{\sigma+ir} \Gamma(s)(Mf)(s)x^{-s}ds, \phi(x)\rangle \to \langle f(x), \phi(x)\rangle \ as \ r \to \infty,$$

for every $\phi \in S$ such that for some $\varepsilon > 0$ supp $\phi \subset [\varepsilon, \infty)$.

Proof. Let $f = F^{(m)}$, where F is a continuous function such that supp $F \subset [0, \infty)$. Take $\sigma_0 \in \mathbf{R}_+$, such that for some k, $F^{(\alpha)} \in S'_k, \alpha = 0, 1, \ldots, m,$ and $\varphi_{s-m} \in S_k$ if Res $> \sigma_0$.

Let $\phi \in S$ and supp $\phi \subset [\varepsilon, \infty)$. For sufficiently large σ_0 and $\sigma > \sigma_0$

$$\frac{1}{2\pi i} \int_{\sigma-ir}^{\sigma+ir} M_0(F(x)e^{-x/2})(s-j)x^{-(s-j)}ds, \quad x >' \varepsilon,$$

converges to $F(x)e^{-x/2}$ in $L^2(\varepsilon, \infty)$ - norm when $r \to \infty$. If follows from [8, p. 94, Theorem 71]. So, by (3) we have

$$\lim_{r\to\infty} \langle \frac{e^{x/2}}{2\pi i} \int_{\sigma-ir}^{\sigma+ir} \Gamma(s)(Mf)(s)x^{-s}ds, \phi(x)\rangle =$$

$$\lim_{r\to\infty} 2^{-m} \langle \sum_{j=0}^{m} \binom{m}{j} \frac{(-2)^j}{2\pi i} \int_{\sigma-ir}^{\sigma+ir} (s-1)\ldots(s-j)$$

$$M_0(Fe^{-x/2})(s-j)x^{-s}ds, e^{x/2}\phi(x)\rangle$$

$$= \lim_{r\to\infty} 2^{-m} \sum_{j=0}^{m} \binom{m}{j} \frac{(-2)^j}{2\pi i} (-1)^j \langle \int_{\sigma-ir}^{\sigma+ir}$$

$$(M_0(Fe^{-x/2})(s-j)x^{-(s-j)}ds)^{(j)}, e^{x/2}\phi(x)\rangle$$

$$= \lim_{r\to\infty} \langle 2^{-m} \sum_{j=0}^{m} \binom{m}{j} \frac{(-2)^j}{2\pi i} \int_{\sigma-ir}^{\sigma+ir}$$

$$(M_0(Fe^{-x/2})(s-j)x^{-(s-j)}ds), (e^{x/2}\phi(x))^{(j)}\rangle =$$

$$= (-1)^m \langle F, (e^{-x/2}e^{x/2}\phi)^{(m)}\rangle = \langle F^{(m)}, e^{-x/2}e^{x/2}p\rangle =$$

$$= \langle 2^{-m} \sum_{j=0}^{m} \binom{m}{j} (-2)^1 \langle (Fe(x)e^{-x/2}, e^{x/2}\phi^{(j)}(x)\rangle = \langle f(x)e^{-x/2}, e^{x/2}\phi(x)\rangle.$$

4. The space of $\mathbf{N}_0$ - transforms

Let us denote by N the space of the Newton series of the form $F = \sum_{n=0}^{\infty} (-1)^n b_n \left(\begin{array}{c} s-j \\ n \end{array} \right)$ where $b_j, j \in \mathbf{N}_0$, are complex numbers. This is the well - known space ([4, Ch.2]). The abscissa of common convergence of an F is denoted by λ_F and the abscissa of ablosute one is denoted by μ_F. The function F is an analytic function in the domain $\text{Re} s > \lambda_F, s \in \mathbf{C}$.

So, M-transform mapps S'_+ onto the space of all $F \in N$ of the give form, such that form some $k \in \mathbf{R}, \sum_{n=1}^{\infty} |b_n|^2 n^{-2k} < \infty$. We denote this space by N_0. Clearly, for any $F \in N_0 - \infty \leq \mu_F < \infty$.

Connections between the spaces of the Newton and Dirichlet series gives ([4, Ch. 2, ℵ 2]) that for an F on the give form

$$\mu_F = \overline{\lim_{n\to\infty}} \frac{\ln \sum_{k=1}^{n} |b_k|}{\ln n} \text{ if } \mu_F \geq 0, \quad \mu_F = \overline{\lim_{n\to\infty}} \frac{\ln \sum_{k=n}^{\infty} |b_k|}{\ln n} \text{ if } \mu_F < 0.$$

This gives the following structural characterization of the space S'_+.

Proposition 7. *Let $f \in S'_+$ and $F = Mf$ be f of the given form. Then*

i. $f \in S'_k \Rightarrow k \geq \mu_F - 1/2$;

ii. $f \not\ni S'_k \Rightarrow \leq \mu_F$:

iii. $k > \mu_F \Rightarrow f \in S'_k$;

iv. $k < \mu_F - \frac{1}{2} \Rightarrow f \not\ni S'_k$.

Proof. With the notation as in Proposition 5 we have from this proposition that $k + 1/2 \geq \mu_F$. So, we get (i). (iv) follows from (i), (ii) from (iii), and (iv) follows from the inequality which holds for some C and n_0:

$$\sum_{n>n_0} |b_n|^2 n^{-2k} < \sum_{n>n_0} |b_n| n^{-k} < C \sum_{n>n_0} b_n| \left(\begin{array}{c} s-1 \\ n \end{array} \right).$$

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REZIME

LAGEROVA EKSPANZIJA U S'_+ I MIDIFIKOVANA MELINOVA TRANSFORMACIJA

U prvom delu rada je proučavan prostor uopštenih funkcija čiji elementi imaju razlaganje po ortonormiranoj Lagerovoj bazi prostora $L^2(0,\infty)$. Taj prostor uopštenih funkcija je u stvari prostor S'_+ i koristeći se ortogonalnim razlaganjima u drugom delu rada je definisana i ispitana modifikovana Melinova transformacija na S'_+. Dve inverzione formule za tu transormaciju su date.

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