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MANY DIMENSIONAL LYAPUNOV CONVEXITY THEOREM FOR AN E_λ-DECOMPOSABLE MEASURE

Endre Pap
Institute of Mathematics, University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

A Lyapunov type theorem on the convexity of the range for a many dimensional E_λ-decomposable measure with respect to a non-strict t-conorm $E_\lambda, 0 < \lambda \leq 1$, is proved.

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1. Introduction

One of the important results in the investigation of ranges of measures is Lyapunov's theorem on compactness and convexity. This theorem has many generalizations ([2], [3], [4], [6], [13]) and many applications, specially in the mathematical economy ([3], [5]).

We shall continue in this paper investigations on E_λ -decomposable measures ([9] - [12], see also [1], [4], [14] - [17]). As a direct consequence of our earlier results [10] we shall formulate in this paper a one-dimensional version of Lyapunov 's theorem for an atomless $\sigma - \perp$- decomposable measure. M.Fedrizzi, M.Squllante and A.G.S.Ventre [4] have proved a version of a one- dimensional Lyapunov theorem for strongly continuous $\perp$-decomposable measures with respect to the Archimedean t-conorm. $\perp$. The closedness part of Lyapunov's theorem cannot be generalized for the many dimensional case even for additive set functions (charges)-see [13], Example 11.4.8. In this paper we shall generalize the convexity part of Lyapunov's theorem for many dimensional $E_\lambda-$ decomposable measures with respect to the non-strict t-conorm $E_\lambda, 0 < \lambda \leq 1$.

2. One dimensional Lyapunov theorems

An operation $\perp : [0,1] \times [0,1] \to [0,1]$ is said to be a t-conorm if it is commutative, associative,nondecreasing in each argument and has zero as a neutral element.

An operation $\top : [0,1] \times [0,1] \to [0,1]$ is said to be a t-norm if it is commutative, associative, nondecreasing and has 1 as a unit.

A t-conorm $\perp$ is said to be partially continuous if it satisfies condition

(F) For each $\varepsilon > 0$ and each $x \in [0,1], x < \varepsilon$, there exists $y \in (0,1]$ such that $x \perp y < \varepsilon$ (see E.Pap [10]).

A t-conorm $\perp$ is said to be Archimedean if $\perp$ is continuous and $\perp(x,x) > x$ for all $x \in (0,1)$. An Archimedean t-conorm is said to be strict if $\perp$ is strictly increasing in $(0,1) \times (0,1)$.

Ling's representation theorem [7]: An operation $\perp$ is an Archimedean t-conorm if and only if there exists an increasing and continuous function $g : [0,1] \to [0,1]$ with $g(0) = 0$, such that

$$\perp(x,y) = g^{(-1)}(g(x) + g(y)),$$

where $g^{(-1)}$ is the pseudoinverse of g, given by

$$g^{(-1)}(y) := \begin{cases} g^{(-1)}(y) \text{ for } y \in [0, g(1)] \\ 1, \text{ for } y \in [g(1), \infty] \end{cases}$$

Function g is called an additive generator of $\perp$ and it is unique up to a positive constant factor. If $\perp$ is a non- strict t-conorm with the additive generator g such that $g(1) = 1$, then we shall call g a normed generator.

For any t-conorm $\perp$ we define (S.Weber [16]):

$$b -_{\perp} a := \inf\{y : a \perp y \geq b\}.$$

For any non-strict Archimedean t-conorm $\perp$ with additive generator g

$$b -_{\perp} a = g^{-1}(g(b) - g(a))$$

for $a \leq b$ holds.

Let Σ be a σ-algebra of subsets of a non empty set X. A set function $m : \Sigma \to [0,1]$ with $m(\emptyset) = 0$ will be called a $\perp$-decomposable measure, if

$$m(A \bigcup B) = m(A) \perp m(B)$$

holds for all $A, B \in \Sigma$ such that $A \bigcap B = \emptyset$. A set function $m : \Sigma \to [0,1]$ with $m(\emptyset) = 0$ will be called a $\sigma - \perp -$ decomposable measure if it holds that

$$m(\bigcup_{n=1}^{\infty} A_n) = \perp_{n=1}^{\infty} m(A_n)$$

for each sequence (A_n) of pairwise disjoint elements from Σ.

A set function $m : \Sigma \to [0,1]$ is order continuous if $\lim_{n \to \infty} m(E_n) = 0$ for any sequence $(E_n), E_n \in \Sigma \quad (n \in N)$, such that $E_n \searrow \emptyset$.

A set $A \in \Sigma$ is an atom of a $\perp$-decomposable measure m iff $m(A) > 0$ and either $m(A \bigcap B) = 0$ or $m(A \setminus B) = 0$ for any $B \in \Sigma$. A set function $m : \Sigma \to [0,1]$ is atomless, if it is without atoms.

As a consequence of Theorem 1.3.4. (the Darboux property) from [10], we obtain the following version of Lyapunov's theorem.

Theorem 1. *Let $m : \Sigma \to [0,1]$ be an atomless order continuous $\sigma - \perp$-decomposable measure with respect to a t-conorm $\perp$, which is partially continuous. Then the range of m is the interval* $[0, m(X)]$.

For the second version of Lyapunov's theorem we need the following definition: a $\perp$-decomposable measure $m : \Sigma \to [0,1]$ is strongly continuous on Σ if for every $\varepsilon > 0$ there exits a finite partition

$$\{A_1, A_2, ..., A_n\}, A_i \in \Sigma \quad (i = 1, 2, ...n),$$

of X, such that

$$m(A_i) < \varepsilon$$

for $i = 1, 2, ..., n$. Then we have, according to [4](Theorem 4.1)

Theorem 2. *Let $m : \Sigma \to [0,1]$ be a $\perp$- decomposable measure with respect to the Archimedean t-conorm $\perp$. If m is strongly continuous, then range $R(m)$ of m is the interval $[0, m(X)]$.*

3. The many dimensional case

Let E_λ for a fixed $\lambda, 0 < \lambda \leq 1$, be defined in the following way

$$xE_\lambda y := \min((x^\lambda + y^\lambda)^{1/\lambda}, 1) \quad (x, y \in [0,1]).$$

Then E_λ is a non-strict t-conorm with the normed additive generator $g(x) = x^\lambda$.

The restricted distribution law

$$c.(xE_\lambda y) = (c.x)E_\lambda(c.y)$$

is valid for all $c \in [0,1]$ and for all $(x, y) \in D$, where

$$D = \{(x, y) : x^\lambda + y^\lambda \leq 1\}$$

(sec. S.Weber [17] and Schwyhla [15]).

A subset C of $[0,1]^n$ is $E_\lambda-$ convex if for each $(x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n) \in C$ such that $((x_1, y_1), (x_2, y_2), ..., (x_n, y_n)) \in D^n$ and each $t \in [0,1]$, we have $(z_1, z_2, ..., z_n) \in C$, where

$$z_i = (c.x_i)E_\lambda((1 -_\perp t).y_i)$$

for $i = 1, 2..., n$. The operation $-_\perp$ is with respect to E_λ. The E_λ-dyadic numbere from [0,1] is of the form

$$e = \frac{a_1}{2} E_\lambda \frac{a_2}{2^2} E_\lambda ... E_\lambda \frac{a_k}{2^k},$$

where $a_i, i = 1, 2, ..., k$, are zero or one.

Theorem 3. *Let $m_1, m_2, ..., m_n$ be strongly continuous E_λ-decomposable measures for a fixed real number $\lambda, 0 < \lambda \leq 1$ defined on a σ-algebra Σ of subset of a set X. Then for the range*

$$R(m_1, m_2, ..., m_n) = \{(m_1(A), m_2(A), ..., m_n(A)) : A \in \Sigma\}$$

of $m_1, m_2, ..., m_n$ the set $D_n \bigcap R(m_1, m_2, ..., m_n)$ is an E_λ-convex subset of the n-dimensional interval $[0,1]^n$, where $D_n = \{(x_1, x_2, ..., x_n) : x_1^\lambda + x_2^\lambda + ... + x_n^\lambda \leq 1\}$.

Proof. Let

$$\mu_i = m_i E_\lambda m_{i+1} E_\lambda ... E_\lambda m_n$$

for $i = 1, 2, ..., n$. We have that the set $D_n \cap R(m_1, m_2, ..., m_n)$ is E_λ - convex if and only if the range $(\mu_1, \mu_2, ..., \mu_n)$ of $\mu_1, \mu_2, ..., \mu_n$ is $E_\lambda-$ convex. So we shall prove that $R(\mu_1, \mu_2, ..., \mu_n)$ is $E_\lambda-$ convex.

For that purpose we shall show by inducition the following proprty of $\mu_1, \mu_2, ..., \mu_n$:

(a) For every $A \in \Sigma$ there exists a set B in Σ, such that $B \subset A$ and $\mu_i(B) = \frac{1}{2}\mu_i(A)$ for $i = 1, 2, ..., n$.

Then (a) will imply that for any arbitrary but fixed set A from Σ and every E_λ- dyadic number e from the interval $[0,1]$, there exists a set A_e such that:

$$A_o = \emptyset, \quad A_1 = A, \quad A_{e_1} \subset A_{e_2}$$

whenever for the $E_\lambda-$ dyadic numbers e_1 and $e_2, 0 \leq e_1 \leq 1$ holds and

$$\mu_i(A_e) = e\mu_i(A) \tag{1}$$

for $i = 1, 2, ..., n$.

Namely, by repeated application of (a) for $A \in \Sigma$ and $k \in N \bigcup \{0\}$, there exists the set $A_{\frac{1}{2^k}}$ such that

$$\mu_i(A_{\frac{1}{2^k}}) = \frac{1}{2^k}\mu_i(A)$$

for $i = 1,2,...,n$. Using the sets $A_{\frac{1}{2^k}}$ we can construct the set A_e for the E_λ - dyadic number $e = \frac{a_1}{2} E_\lambda \frac{a_2}{2^2} E_\lambda ... E_\lambda \frac{a_s}{2^s}$, where $a_i, i = 1,2,...,s$, is zero or one. For example, if

$$e = \frac{1}{2} E_\lambda \frac{1}{2^2},$$

then $A_e = A \backslash A_{\frac{1}{2^2}}$ and

$$\mu(A \backslash A_{\frac{1}{2^2}}) = (1 -_\perp \frac{1}{2^2}) \mu_i(A) = (\frac{1}{2} E_\lambda \frac{1}{2^2}) \mu_i(A)$$

for $i = 1,2,...,n$.

Further, for any real number t from $[0,1]$ we take all the E_λ-dyadic numbers e such that $e \leq t$. Then we have $t = \sup\{e : e \leq t\}$. Therefore, if we take

(2) $$A_t = \bigcup_{e \leq t} A_e,$$

where the union runs through the E_λ - dyadic numbers $e \leq t$, we have $A_t \in \Sigma$ and

(3) $$\mu_i(A_t) = t.\mu_i(A)$$

for $i = 1,2,...,n$.

Now we shall prove (a) by induction. For $n = 1, (a)$ follows by Theorem 2.2. Let us assume (a) to be true for $n = k$ and prove (a) for $n = k + 1$. Let

$$\mu_i = m_i E_\lambda m_{i+1} E_\lambda ... E_\lambda m_{k+1}$$

for $i = 1,2,...,k + 1$, where $m_1, m_2, ..., m_{k+1}$ are strongly continuous E_λ-docomposable measures.

Let $B' \in \Sigma$ be a set obtained by the induction step for $n = k$, i.e. B' is a subset of A such that

$$\mu_i(B') = \frac{1}{2} \mu_i(A)$$

for $i = 1,2,...,k$. If it is

$$\mu_{k+1}(B') = \frac{1}{2} \mu_{k+1}(A),$$

then we have proved the desired equality for $n = k + 1$. If not, then it is either.

$$\mu_{k+1}(B') < \frac{1}{2}\mu_{k+1}(A) < \mu_{k+1}(A \setminus B')$$

or

$$\mu_{k+1}(A \setminus B') < \frac{1}{2}\mu_{k+1}(A) < \mu_{k+1}(B').$$

We shall suppose the first case. For the second case we proceed in an analogous way. By the induction hypothesis we take for $\mu_1, \mu_2, ..., \mu_k$ the sets B'_t $(0 \leq t \leq 1)$ and $(A \setminus B')_t$ $(0 \leq t \leq 1)$ which are corresponding sets to the sets B' and $A \setminus B'$, respectively, and defined by (2) such that satisfy (3). For any two real numbers a and b such that $b \leq a$, we have

$$\mu_{k+1}(B'_a) -_{\perp} \mu_{k+1}(B'_b) = \mu_{k+1}(B'_a \setminus B'_b) \leq \mu_k(B'_a \setminus B'_b) = \mu_k(B'_a -_{\perp} \mu_k(B'_b) =$$

$$(a -_{\perp} b)\mu_k(B'),$$

and analogously

$$\mu_{k+1}((A \setminus B')_a) -_{\perp} \mu_{k+1}((A \setminus B'_b) \leq (a -_{\perp} b)\mu_k(A \setminus B').$$

Since the operation $-_{\perp}$ is continuous, we obtain by the preceding inequalities that the functions $\mu_{k+1}(B'_t)$ and $\mu_{k+1}((A \setminus B')_t)$ are continuous with respect to t.

Namely, if an arbitrary but fixed sequence (x_n) from $[0,1]$ converges to $c \epsilon [0,1]$, then $x_n - c \to 0$ implies $x_n -_{\perp} c \to 0$. The last inequalities imply

$$\mu_{k+1}(B'_{x_n}) -_{\perp} \mu_{k+1}(B'_c) \to 0,$$

i.e.

$$\mu_{k+1}(B'_{x_n})^{\lambda} -_{\perp} \mu_{k+1}(B'_c)^{\lambda} \to 0.$$

Hence,

$$\mu_{k+1}(B'_{x_n}) \to \mu_{k+1}(B'_c),$$

i.e. $\mu_{k+1}(B'^t)$ is continuous. This implies that the function

$$F(t) := \mu_{k+1}(B'_t \bigcup (A \setminus B')_{1-_{\perp}t} = \mu_{k+1}(B'_t) E_{\lambda} \mu_{k+1}((A \setminus B')_t)$$

is also a continuous function of t for $0 \leq t \leq 1$. Then F takes all the values between $F(1) = \mu_{k+1}(B')$ and $F(0) = \mu_{k+1}(A \setminus B')$. So there exists $t_o \in [0,1]$ such that

$$F(t_o) = \frac{1}{2}\mu_{k+1}(A).$$

Taking $B = B'_{t_o} \bigcup (A \setminus B')_{1-_\perp t}$, we find the desired set in (1) for $n = k + 1$.Namely, we also have for each i between 1 and k

$$\mu_i(B) = \mu_i(B'_{t_o}) E_\lambda \mu_i((A \setminus B')_{1-_\perp t_o}) =$$
$$= t_o \mu_i(B') E_\lambda (1 -_\perp t_o) \mu_i(A \setminus B') =$$
$$= t_o, \frac{1}{2}.\mu_i(A) E_\lambda (1 -_\perp t_o).\frac{1}{2}\mu_i(A) =$$
$$= (t_o E_\lambda (1 -_\perp t_o)).\frac{1}{2}.\mu_i(A) = \frac{1}{2}\mu_i(A).$$

Now, we can prove that $R(\mu_1, \mu_2, ..., \mu_{k+1})$ is E_λ-convex. Taking $T, S \in \sum$ and $0 \leq t \leq 1$ we have

$$t.\mu_i(T) E_\lambda (1 -_\perp t) \mu_i(S) =$$
$$= (t.\mu_i(T \setminus S)) E_\lambda (t.\mu_i(T \bigcap S)) E_\lambda ((1 -_\perp t) \mu_i(S \setminus T)) E_\lambda ((1 -_\perp t).\mu_i(S \setminus T)) =$$
$$= (\mu_i((T \setminus S)_t)) E_\lambda (\mu_i((S \setminus T)_{1-_\perp t})) E_\lambda (\mu_i(S \cap T)) =$$
$$\mu_i((T \setminus S)_t \cup (S \cap T) \cup (S \setminus T)_{1-_\perp t})$$

for every $i = 1, 2, ..., k + 1$.

Therefore we have proved our theorem by induction.

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REZIME

VIŠEDIMENZIONALNA TEOREMA LJAPUNOVA O KONVEKSNOSTI ZA E_λ - DEKOMPOZABILNU MERU

U radu se dokazuje teorema tipa Ljapunova o konveksnosti skupa vrednosti višedimenzionalne E_λ-dekompozabilne mere u odnosu na t-konormu $E_\lambda, 0 < \lambda \leq 1$.

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