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SOME COMMUTATIVITY CONDITIONS FOR RINGS

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Abstract

Earlier results concerning the commutativity of rings with unity due to Quadri and Khan (Math. Japon., 33(2) (1988), 275-279) and Psomopoulos (Math. Japon., 29 (3) (1984), 371-373) have been obtained under a different set of conditions. The method of proof is based on an iteration technique.

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1. Introduction

Recently, Psomopoulos [3], Quadri and Khan [4] proved some interesting commutativity theorems for rings with unity. While the work of Quadri and Khan [4] generalized a famous result due to Bell [1], the result of Psomopoulos [3] is new in its own right.

In this paper, we shall present some results on the commutativity of rings using polynomial identities as considered by the above authors but under a different set of conditions and employing an entirely new technique of proof which was given earlier by Tong [5].

Throught this note, R stands for a ring with unity 1. As usual, we write $[x, y] = xy - yx$, where x and y are arbitrary elements from R. The following well-known results will be frequently used in the sequel.

Lemma 1.1 *([2]). Let R be a ring with unity 1 and $f : R \to R$ be a function such that $f(x+1) = f(x)$, for every x in R. If for some positive integer $n, x^n f(x) = 0$ for all x in R or $f(x)x^n = 0$ for all x in R, then necessarily $f(x) = 0$.*

Lemma 1.2 *([5]). Let R be a ring with unity 1. Let $I_0^r(x) = x^r$. If $k > 1$, let $I_k^r(x) = I_{k-1}^r(x+1) - I_{k-1}^r(x)$. Then, $I_{r-1}^r(x) = \frac{1}{2}(r-1)r! + r!x; I_r^r(x) = r!$, and $I_j^r(x) = 0$ for $j > r$.*

2. Results

In a recent paper, Wei Zong Xuan [6] proved that a semi prime ring R is commutative if for all x, y, z in R any one of the following conditions is satisfied

(i) $[x^2y^2 - xy^2x, z] = 0$,

(ii) $[x^2y^2 - yx^2y, z] = 0$.

Two of our results in this section are motivated by the above polynomial identities. Our conditions are in fact borrowed from a pre-print entitled " *On a commutativity condition for rings* " by M.A. Quadri et. al.

Theorem 2.1 *Let R be a ring with unity having the property: "there exist positive integers m and n such that*

$$(A) \qquad [x^m y^n - xy^n x, x] = 0$$

holds for all x, y in R." If every commutator in R is $2(n!)(m!)$ - torsion free, then R must be commutative.

Proof. We can write condition (A) of the theorem as

$$(1) \qquad x^m[x, y^n] - x[x, y^n]x = 0.$$

If $n = 1$, then (1) gives

(1') $$x^m[x,y] = x[x,y]x.$$

If $n > 1$, firstly, we shall apply iteration on y^n in (1). To do this, let $I_j(y) = I_j^n(y)$, for $j = 0,1,2,3,\ldots$ Then (1) can be expressed as

(2) $$x^m[x,I_0(y)] - x[x,I_0(y)]x = 0.$$

Let us put $y = y+1$ in (2). Then, we obtain

$$x^m[x,I_0(y+1)] - x[x,I_0(y+1)]x = 0.$$

Now,using Lemma 1.2, we have

$$x^m[x,I_1(y)+I_0(y)] - x[x,I_1(y)+I_0(y)]x = 0.$$

Using (2), we obtain

(3) $$x^m[x,I_1(y)] - x[x,I_1(y)]x = 0.$$

Again, letting $y = y+1$ in relation (3) and using Lemma 1.2, we get

(4) $$x^m[x,I_2(y)] - x[x,I_2(y)]x = 0.$$

Finally, replacing y by $y+1$ in (4) and then iterating $(n-1)$ times, we obtain

$$x^m[x,I_{n-1}(y)] - x[x,I_{n-1}(y)]x = 0.$$

But by Lemma 1.2, $I_{n-1}(y) = (n!)y + \frac{1}{2}(n-1)n!$. So,

(5) $$(n!)\{x^m[x,y] - x[x,y]x\} = 0,$$

and (1') follows immediately from (5).

If $m > 1$, we shall iterate x^m. To achieve it, we let $I_0(x) = x^m$ in (1') which then becomes

$$\{I_0(x)[x,y] - x[x,y]x\} = 0.$$

Let us replace x by $x+1$ in the above expression getting thereby

$$\{I_0(x+1)[x,y]-(x+1)[x,y](x+1)\}=0.$$

Then, as before, we get

$$\{I_1(x)[x,y]-[x,y]-[x,y]x-x[x,y]\}=0. \tag{6}$$

Now, replacing x by $x+1$ in (6) and using (6), we get

$$\{I_2(x)[x,y]-2[x,y]\}=0. \tag{7}$$

Again, putting $x=x+1$ in (7), we have

$$\{I_2(x+1)[x,y]-2[x,y]\}=0,$$

which when combined with (7) gives

$$\{I_3(x)[x,y]\}=0. \tag{8}$$

Similarly, we can get

$$\{I_4(x)[x,y]\}=0. \tag{9}$$

Finally, letting $x=x+1$ and iterating m times, we are left with

$$\{I_m(x)[x,y]\}=0.$$

But by Lemma 1.2, $I_m(x)=m!=I_m^m(x)$. So

$$(m!)[x,y]=0 \quad \text{i. e.} \quad [x,y]=0.$$

If $m=1$, from (1') by iteration on x we get

$$x[x,y]+[x,y]x=0,$$

and then

$$z[x,y]=0 \quad \text{i. e.} \quad [x,y]=0.$$

Therefore, R is commutative. This completes the proof.

Now, we shall present a short and easy proof of a result due to Quadri and Khan [4] under some extra torsion condition on commutators in the ring.

Theorem 2.2 *Let m and n be two fixed positive integers and R a ring with unity in which every commutator is $(m!)(n!)$ - torsion free. If the condition*

$$\text{(B)} \qquad [xy - y^m x^n, x] = 0$$

holds for all x, y in R, then R must be commutative for $m > 1$ and $n > 0$, and for $m > 0$ and $n > 1$.

Proof. Condition (B) is equivalent to

$$\text{(1)} \qquad x[x, y] = [x, y^m]x^n.$$

If $m > 1$, as in the proof of Theorem 2.1, we let $I_0(y) = y^m$.
Then (1) can be re-written as

$$\text{(2)} \qquad x[x, y] = [x, I_0(y)]x^n.$$

Now, letting $y = y + 1$ in the above expression and using (2), we get

$$[x, I_1(y)]x^n = 0.$$

A similar argument can be used to get

$$\text{(3)} \qquad [x, I_2(y)]x^n = 0.$$

Finally, replacing y by $y + 1$ in (3) and iterating $(m - 1)$ times, we obtain

$$(m!)[x, y]x^n = 0,$$

or equivalently

$$(m!)[x, yx^n] = 0.$$

As every commutator in R is $(m!)$ - torsion free, we get

$$[x, yx^n] = [x, y]x^n = 0,$$

and from there it follows that $[x, y] = 0$ because of Lemma 1.1

If $m = 1$ and $n > 1$, then (1) becomes

$$x[x, y] = [x, y]x^n,$$

and from there by iteration on x we get

$$(n!)[x, y] = 0 \quad \text{i. e.} \quad [x, y] = 0.$$

Thus R is commutative.

Following the idea developed in the proofs of our Theorem 2.1 and Theorem 2.2, one can supply another proof of a result due to Psomopoulos [3]. Here we shall only sketch of the proof for the sake of completeness.

Theorem 2.3 *Let $m > 1$ and $n > 0$ be two fixed positive integers and R a ring with unity in which every commutator is $(m!)$ - torsion free. If the condition*

(C) $$[x^n y - y^m x, x] = 0$$

holds for all x, y in R, then R must be commutative.

Proof. The equivalent form of condition (C) is given by

$$x^n[x, y] - [x, y^m]x = 0.$$

As in the proofs of the previous two results, firstly the iteration is applied to y^m by putting $y = y + 1$ to get

$$[x, I_1(y)]x = 0.$$

Now, as before, we get

$$(m!)[x, y]x = 0,$$

which along with Lemma 1.1 for $f(x) = [x, y]$ gives $[x, y] = 0$. Hence R is commutative.

Finally, we shall present a result which is similar to our Theorem 2.1.

Theorem 2.4 *Let $m > 1$ and $n > 0$ be two fixed positive integers and R a ring with unity in which every commutator is $2(n!)(m!)$ - torsion free and the condition*

$$(D) \qquad [x^m y - xy^n x, y] = 0$$

holds for all x, y in R. Then R must be commutative.

Proof. Let $m \neq 2$. We set $I_0(y) = yn$. Then (D) becomes

$$[x^m I_0(y) - xI_0(y)x, y] = 0.$$

Now, replacing y by $y + 1$ in the above expression and using the same expression again, we get

$$[x^m I_1(y) - xI_1(y)x, y] = 0.$$

Now, iterating the above expression n times we obtain

$$[x^m I_n(y) - xI_n(y)x, y] = 0.$$

Hence,

$$(n!)[x^m - x^2, y] = 0.$$

As every commutator in R is $(n!)$ - torsion free, we get

$$[x^m - x^2, y] = 0, \quad \text{i. e.} \quad [x^m, y] = [x^2, y].$$

It is not hard by iteration on x to have the last expression above reduced to

$$(m!)[x, y] = 0 \quad \text{if} \quad m > 2, \quad \text{and} \quad 2[x, y] = 0 \quad \text{if} \quad m = 1.$$

Once, again the hypothesis on commutators yields $[x, y] = 0$. For $m = 2$ and $n > 1$ from (D) we get by iteration on X:

$$[2xy^n - xy - yx, y] = 0,$$

i. e.

$$2 \cdot [x, y]y^n = [x, y]y + y[x, y],$$

and from there by iteration on y :

$$2 \cdot (n!)[x,y] = 0 \quad \text{i. e.} \quad [x,y] = 0.$$

Thus is all cases, R must be commutative.

Remarks.

i. Psomopoulos [3] assumed R to be s-unital ring. All previous theorems are still true for R as an s-unital ring. Namely, for arbitrary elements x, y of an s-unital ring R there exists $e = e(x,y)$ in R such that $ex = xe = x$ and $ey = ye = y$. With $e = e(x,y)$ instead of 1, Lemma 1.1 and Lemma 1.2 with $I_r^{r-1} = \frac{1}{2}(r-1)r!e + r!x$ and $I_r^r(x) = r!e$, obviously are still true for R as an s-unital ring. This permits our iteration on x or on y by setting $x + e$ or $y + e$ for x or y respectively.

ii. In the preprint of M. A. Quadri et. al, mentioned in the beginning of this section, the ring under consideration was semi-prime satisfying either of the conditions (A) or (D).

iii. Concering Theorem 2.1 and Theorem 2.4, we note that for $m > 1$ or $n > 1$, 2 divides $(m!)(n!)$ and then every commutator in R is $2 \cdot (m!)(n!)$ - torsion free if and only if every commutator in R is $(m!)(n!)$ - torsion free.

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REZIME

NEKI USLOVI KOMUTATIVNOSTI ZA PRSTENE

Neki rezultati komutativnosti Quadrija i Khana (Math. Japon., 33(2)(1988), 275-279) i Psomopoulosa (Math. Japon., 29(3)(1984), 371-373) ovde su dobijeni pod drugim uslovima. Naš metod dokazivanja se zasniva na jednoj iteracionoj tehnici.

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