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ON A METHOD FOR SOLVING AN ALMOST LINEAR SYSTEM OF EQUATIONS

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Abstract

In order to solve some special classes of nonlinear systems of equations the Modified Accelerated Overrelaxation - Newton (MAORN) method, introduced in [1], is used. The same nonlinear systems have been considered by Chernyak, [5] . For their numerical solution he has used the AORN (Accelerated Overrelaxation - Newton) method. By using a similar technique as in Chernyak's papers [4] and [5], some sufficient conditions are obtained for the global convergence. The modified method and the convergence result are simpler than the ones from [5] .

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1. Introduction

A system of nonlinear equations

$$f_i(x_1, x_2, ..., x_n) = 0, \ i \in N := \{1, 2, ..., n\},$$

will be solved by using the MAORN (Modified Accelerated Overrelaxation - Newton) method: for $i \in N$, $k = 0, 1, 2, \ldots$

$$x_i^{k+1} = x_i^k - \omega \Delta_i^{(k)}(\sigma), \quad \bar{x}_i^{k+1} = x_i^k - \sigma \Delta_i^{(k)}(\sigma),$$

$$\Delta_i^{(k)}(\sigma) = \frac{f_i(\bar{x}_1^{k+1}, \dots, \bar{x}_{i-1}^{k+1}, x_i^k, \dots, x_n^k)}{d_i(\bar{x}_1^{k+1}, \dots, \bar{x}_{i-1}^{k+1}, x_i^k, \dots, x_n^k)},$$

where σ, $\omega \in \mathbf{R}$, $\omega \neq 0$, and $d_i(x)$, $i \in N, x \in \mathbf{R}^n$ are arbitrary real nonzero functions. This method was introduced in our paper [1].

In the case when $d_i(x) = \partial f_i/\partial x_i$, $i \in N$, $(x = [x_1, x_2, \dots, x_n]^T)$, the MAORN method reduces to an AORN (Accelerated Overrelaxation - Newton) one, investigated by Chernyak in [5].

It is obvious that the MAORN method has one very important advantage: we do not have to calculate the partial derivatives of the given nonlinear functions. Because of that we have started to investigate this method. But, in convergence analysis, until now, we have obtained sufficient conditions for local convergence only. Also, these sufficient conditions depend on the value of the Jacobian matrix at the exact solution of our nonlinear system.

The aim of this paper is to give some sufficient conditions for the global convergence of the MAORN method, at least for some specias nonlinear systems.

We shall analyse the same class of nonlinear systems as in [5], i.e. nonlinear systems of the following type:

$$(1) \qquad f_i(x_1, x_2, \dots, x_n) \equiv \sum_{j=1}^{n} a_{ij} x_j + g_i(x_i) - b_i = 0, \; i \in N,$$

where a_{ij}, $b_i \in \mathbf{R}$, $i, j \in N$, $a_{ii} \neq 0$, $i \in N$, and $g_i(t)$, $t \in \mathbf{R}$, are real nonlinear differentiable functions. For nonzero functions $d_i(x)$ we shall choose

$$d_i(x) = a_{ii}, \; i \in N, \; x \in \mathbf{R}^n,$$

so that the MAORN method for solving system (1) will have the following form:

$$\text{for } i \in N, \; k = 0, 1, 2, \dots$$

$$(2) \qquad x_i^{k+1} = x_i^k - \omega \Delta_i^{(k)}(\sigma), \quad \bar{x}_i^{k+1} = x_i^k - \sigma \Delta_i^{(k)}(\sigma),$$

$$\Delta_i^{(k)}(\sigma) = \frac{f_i(\bar{x}_1^{k+1}, \dots, \bar{x}_{i-1}^{k+1}, x_i^k, \dots, x_n^k)}{a_{ii}}.$$

Obviously, it can be applied without knowledge of partial derivatives of the nonlinear functions g_i, $i \in N$. Note that in paper [5] Chernyak used the

AORN method, which in this special case has the same form as (2), except that for a_{ii} in each iteration we should take

$$a_{ii} + g_i'(x_i^k).$$

In order to obtain some sufficient conditions for the convergence (global, if it is possible) of the MAORN method (2), we shall use the result of Theorem 1 from [4]. We are going to describe this result in the next section.

2. Preliminaries

Let $\Phi(x,y)$ be a continuous mapping from $X \times X$ to X (X is a Banach space). For the numerical solution of

$$\Phi(x,x) = 0, \tag{3}$$

the folloving iterative method is used:

$$x^{k+1} = x^k - (\Phi_x'(x^k,x^k))^{-1}\Phi(x^k,x^k), \quad k = 0,1,\ldots, \tag{4}$$

where $\Phi_x'(x,y)$ is an F-derivative of Φ with respect to the first argument.

Theorem 1. [4] *Let the following conditions be satisfied:*

1. *For each* $x \in S := \{x \in X|\ \|x - x^0\| \le r\}$ *there exists the linear inverse operator* $\Gamma(x) = [\Phi_x'(x,x)]^{-1}$ *and* $\|\Gamma(x)\| \le B$,

$$\|\Gamma(x^0)\Phi(x^0,x^0)\| \le \eta_0;$$

2. *For each* $x, y \in S$ *it holds that*

$$\|\Phi_x'(x,y) - \Phi_x'(y,y)\| \le M\|x - y\|,$$

$$\|\Gamma(x)[\Phi(x,x) - \Phi(x,y)]\| <$$
$$< \delta_1\|x - y\| + \delta_2\|\Gamma(x)\Phi(y,y)\|\ \|x - y\| + \delta_3\|x - y\|^2,$$

where $\delta_1,\ \delta_2,\ \delta_3 \ge 0$;

3. $h_1 = \delta_1 + \delta\eta_0 < 1$, *where* $\delta = 0.5BM + \delta_2 + \delta_3$;

4. $r \ge t^* = (1 + \sum_{k=1}^{\infty}\prod_{j=1}^{k} h_j)\eta_0$, *where* $h_j = \delta_1 + \delta\eta_0 \prod_{\nu=1}^{j-1} h_\nu,\ j = 2,3,\ldots.$

Then there exists $x^* \in S^* = \{x \in X \mid \|x - x^0\| \le t^*\}$, so that $\Phi(x^*, x^*) = 0$ and the iterative method (4) converges to x^*.

3. Convergence result

If we intend to apply Theorem 1, we have to choose the continuous nonlinear operator Φ , such that system (3) is equivalent to nonlinear system (1), and the MAORN method (2) is the same as method (4).

It is easy to prove that the nonlinear operator given by

$$(\Phi(x,y))_i = f_i\left(v^{(i)}(y)\right) + \frac{a_{ii}}{\omega}(x_i - y_i),$$

$$v^{(i)}(y) = [v_1(y), \ldots, v_{i-1}(y), y_i, \ldots, y_n]^T, \quad v_i(y) = y_i - \frac{\sigma}{a_{ii}} f_i\left(v^{(i)}(y)\right),$$

satisfies both of the previous conditions.

The following convergence result is some kind of generalization of the Sassenfeld criteria. In the linear case the same result for the AOR method was given in [2]. In the nonlinear case, for the same class of nonlinear systems and for the AORN method, the analogous result was given in [5].

Let us denote

$$p_{1i}(\sigma) = \sum_{j=1}^{i-1} \frac{|a_{ij}|}{|a_{ii}|}(|1-\sigma| + |\sigma| p_{1j}(\sigma)) + \sum_{j=i+1}^{n} \frac{|a_{ij}|}{|a_{ii}|}, \ i \in N;$$

$$p_{2i}(\sigma) = |\sigma| \sum_{j=1}^{i-1} \frac{|a_{ij}|}{|a_{ii}|} p_{2j}(\sigma) + 1, \ i \in N.$$

Theorem 2. *Let*

$$|g_i'(t)| \le \gamma, \ t \in \mathbf{R}, \ i \in N;$$

$$|a_{ii}| \ge a > 0, \ i \in N;$$

$$\delta_1 = \max_{i \in N}\{|1-\omega| + |\omega| p_{1i}(\sigma) + \frac{|\omega|}{a}\gamma p_{2i}(\sigma)\} < 1.$$

Then for each $x^0 \in \mathbf{R}^n$, the MAORN method converges to the solution x^* of system (1) and

$$\|x^* - x^k\|_\infty \le \frac{|\omega| \, \|\Phi(x^k, x^k)\|_\infty}{a(1-\delta_1)}, \ k = 0, 1, 2, \ldots. \tag{5}$$

Proof. It is easy to see that

$$\Phi'_x(x,y) \equiv diag\left(\frac{a_{11}}{\omega},\dots,\frac{a_{nn}}{\omega}\right).$$

Hence, conditions 1) from Theorem 1 are satisfied with $B = |\omega|/a$ and arbitrary x^0 and r. Related η_0 always exists.
The first relation in conditions 2) is satisfied with $M = 0$. The second one, with $\delta_2 = \delta_3 = 0$ follows from relation

$$|\frac{\omega}{a_{ii}}(\Phi_i(x,x) - \Phi_i(x,y))| < \{|1-\omega| + |\omega|p_{1i}(\sigma) + \frac{|\omega|}{a}\gamma p_{2i}(\sigma)\}\|x-y\|_\infty,$$

which can be proved by using the mathematical induction.
Condition 3) reduces to $h_1 = \delta_1 < 1$, and condition 4)

$$t^* = \eta_0/(1-\delta_1) \le r$$

is always satisfied, because r is arbitrary.
From Theorem 1 we conclude that method (4), i.e. the MAORN method converges to x^*, for which is $\Phi(x^*,x^*) = 0$ and

$$\|x^0 - x^*\|_\infty \le t^* = \eta_0/(1-\delta_1) \le \frac{|\omega|\ \|\Phi(x^0,x^0)\|_\infty}{a(1-\delta_1)}.$$

Because all our considerations do not depend on start value x^0 and because each iteration x^k can be considered as a start one, we conclude that error estimation (5) is valid.

□

If we denote

$$p(\sigma) = \max_{i\in N}\{p_{1i}(\sigma) + \frac{\gamma}{a}p_{2i}(\sigma)\},$$

$$l_i = \sum_{j=1}^{i-1}\frac{|a_{ij}|}{|a_{ii}|},\quad u_i = \sum_{j=i+1}^{n}\frac{|a_{ij}|}{|a_{ii}|},\quad i \in N,$$

after some calculations we shall obtain

$$p(\sigma) \le \max_{i\in N}\frac{|1-\sigma|l_i + u_i + \gamma/a}{1-|\sigma|l_i}.$$

Now, it is easy to see that the condition $\delta_1 < 1$ from Theorem 2 can be replaced by the following condition:

$$\max_{i\in N} \frac{|1-\omega| + (|\omega||1-\sigma| - |\sigma||1-\omega|)l_i + |\omega|u_i + |\omega|\gamma/a}{1-|\sigma|l_i} < 1.$$

So, as a direct corollary of Theorem 2 we obtain the following theorem.

Theorem 3. *Let* $1-|\sigma|l_i > 0,\ i \in N;$

$$|g_i'(t)| \leq \gamma,\ t \in \mathbf{R},\ i \in N;$$

$$|a_{ii}| \geq a > 0,\ i \in N;$$

$$\delta^* = \max_{i\in N} \frac{|1-\omega| + (|\omega||1-\sigma| - |\sigma||1-\omega|)l_i + |\omega|u_i + |\omega|\gamma/a}{1-|\sigma|l_i} < 1.$$

Then for each $x^0 \in \mathbf{R}^n$ *the MAORN method converges to solution* x^* *of the system (1) and*

$$\|x^* - x^k\|_\infty \leq \frac{|\omega|\ \|\Phi(x^k, x^k)\|_\infty}{a(1-\delta^*)},\ k = 0, 1, 2, \ldots.$$

Let us remark that in the linear case, when $\gamma = 0$, the MAORN method (2) reduces to the AOR method and δ^* is an upper bound for the maximum norm of the AOR matrix, obtained in [2].

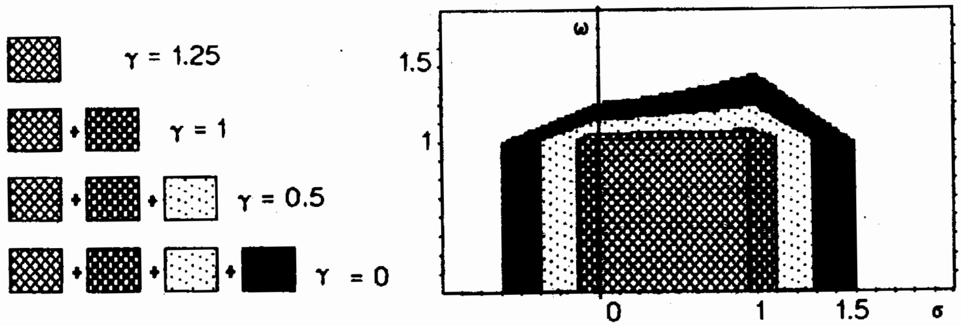
4. Numerical Example

We shall consider a system of nonlinear equations, for which there is

$$l_i = 1/3,\ i \in N \setminus \{1\};\quad u_i = 1/4,\ i \in N\setminus\{n\};$$

$$l_1 = u_n = 0;\quad a = 3.$$

For different nonlinearity (i.e. for different choices of the value γ), in the following picture we can see how to choose the parameters σ and ω, so that $\delta^* < 1$, i.e. the MAORN method converges (Theorem 3).



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REZIME

O JEDNOM POSTUPKU ZA REŠAVANJE SKORO LINEARNIH SISTEMA JEDNAČINA

Za rešavanje nekih specijalnih nelinearnih sistema jednačina koristićemo modifikovani AOR - Njutnov (MAORN) metod, koji smo definisali u radu [1]. Iste nelinearne sisteme posmatrao je Černjak, [5]. Za njihovo numeričko rešavanje on je koristio AORN (AOR - Njutnov) metod. Koristeći sličnu tehniku kao u radovima Černjaka [4], [5], dobićemo neke dovoljne uslove za globalnu konvergenciju. I naš modifikovani metod i rezultati o konvergenciji jednostavniji su od odgovarajućih u radu [5].

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