Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 20, 2 (1990), 91-94

Review of Research
Faculty of Science
Mathematics Series

ARBITRARINESS OF MULTIPLICITY FOR A LINEAR NON – ANTICIPATIVE TRANSFORMATION OF THE WIENER PROCESS

Zoran A. Ivković
Faculty of Mathematics, University of Belgrade
Studentski trg 16, 11000 Beograd, Yugoslavia

Abstract

In the paper the example is given of the process $\{X(t), t > 0\}$ defined by $X(t) = \int_0^t f(t; u)dW(u)(\{W(t)\}$ is Winer process), having an arbitrary finite multiplicity N and a maximal spectral type which is absolutely continuous.

AMS Mathemetics Subject Classification (1980): 60G12
Key words and phrases: Spectral type and multiplicity of second order process.

The notions of the spectral multiplicity theory in the time-domain analysis of continuous second-order processes used in this paper are form the classical papers [1] and [3]. Also, we shall refer to [6].

Let the process $\{X(t), t > 0\}$ be a linear non-anticipative transformation of a standard Wiener process (considered as a wide-sense martingale) $\{W(t), t > 0\}$ i.e.

$$X(t) = \int_0^t f(t; u)dW(u), f(t; \cdot) \in \mathcal{L}_2(du). \quad (1)$$

It is stated in [5] that for the arbitrary chain

$$F_0(t) \succeq F_1(t) \succeq \ldots \succeq F_{N-1}(t) \ (N \text{ may be } \infty) \quad (2)$$

of distribution functions ordered by absolute continuity, there exists the process $\{X(t), t > 0\}$ for which (2) is its spectral type for $t \geq \varepsilon$, where $\varepsilon > 0$ is arbitrary and fixed. We cannot see a way to remove the restriction $t \geq \varepsilon$. Paper [2] (according to [7]) contains the example of the process $\{X(t), t > 0\}$ with the multiplicity $N > 1$, but the spectral type of $\{X(t)\}$ involves discontinuity points.

In this paper we shall give one construction of the process $\{X(t)\}$ with, arbitrary finite multiplicity N and all of whose functions $F_k(t), k = \overline{0, N-1}$ in (2) are absolutely continuous.

Let a partition of $(0,\infty)$ on disjoint sets $S_0, S_1, \dots, S_{N-1}$ be defined by $S_k = \bigcup_{i=-\infty}^{\infty}(2^{k+Ni}, 2^{k+1+Ni}]$ and let $I_k(u), u > 0$, be the indicator function of S_k. Let $I(2^{-k}t; u)$, $u > 0$, be the indicator function of $(0, 2^{-k}t]$. We define the processes $\{Z_k(t),\ t > 0\}$, $k = \overline{0, N-1}$ by

$$(3) \qquad Z_k(t) = \int_0^t I(2^{-k}t; u) I_k(u) dW(u).$$

Proposition 1. *The processe $\{Z_k(t)\}, k = \overline{0, N-1}$ are mutually orthogonal wide-sense martingales. Continuous functions $F_k(t) = \|Z_k(t)\|^2 = EZ_k^2(t), t > 0$, (linearly) increasie on S_0 only.*

Proof. Let $0 < s \leq t$, then $< Z_k(s), Z_k(t) >= EZ_k(s)Z_k(t) =$ $= \int_0^s I(2^{-k}s; u) I(2^{-k}(t); u) I_k^2(u) du = \int_0^s I(2^{-k}s; u) I_k(u) du$ is a function of s only. This means that $\{Z_k(t)\}$ is that wide-sense martingale. Also for $k \neq l$ and arbitrary $s, t > 0$ we have $\langle Z_k(s), Z_l(t)\rangle = \int_0^s I(2^{-k}s; u) I(2^{-k}t; u) I_k(u) I_l(u) du =$ 0 because $I_k(u) I_l(u) = 0$ for $k \neq l$. Finally, from the fact that $t \in S_0$ is equivalent to $2^{-k}t \in S_k, k = \overline{1, N-1}$, we conclude that if $t \in S_0$, then $I(2^{-k}t; u) I_k(u) = 1$ for $u \in (0, 2^{-k}t] \cap S_k$. Hence, $F_k(t)$ linearly increase for $t \in S_0$. If $t \notin S_0$, then $2^{-k}t \notin S_k$ and $I(2^{-k}t; u) I_k(u) = 0$ for all $u > 0$ i.e. $F_k(u)$ is constant on each interval $\bigcap_{k=1}^{N-1}(2^{k+Ni}, 2^{k+1+Ni}], i = \overline{-\infty, \infty}$.

There are several constructions of the continuous process $\{X(t)\}$ having $\{Z_k(t)\}, k = \overline{0, N-1}$ as its innovation process:

$$(4) \qquad X(t) = \sum_{k=0}^{N-1} \int_0^t g_k(t; u) dZ_k(u)$$

Let $\mathcal{H}(Y;t)$ be the mean-square linear closure of $\{Y(u), u \leq t\}$. In the Cramér-Hida representation (4) $\mathcal{H}(X;t) = \bigotimes \sum_{k=0}^{N-1} \mathcal{H}(Z_k;t)$. Since $\mathcal{H}(Z_k;t) \subset \mathcal{H}(W;t)$, we conclude that $\mathcal{H}(X;t) \subset \mathcal{H}(W;t)$ or $X(t) \in \mathcal{H}(W;t)$. So, $\{X(t)\}$ is the non-anticipative transformation (1) of $\{W(t)\}$.

One the simplest constructions of $\{X(t)\}$ is in [4]: Let $\varphi(t), t > 0$, be a continuous function but not absolutely continuous in any interval. Then, $\{X(t), t > 0\}$ defined by $X(t) = \sum_{k=0}^{N-1} \varphi^k(t) Z_K(t)$ is the N-ple Markov process of the multiplicity N. In our construction, $\{X(t)\}$ is the following non-anticipative transformation.

$$X(t) = \int_0^t \{\sum_{k=0}^{N-1} \varphi^k(t) I(2^{-k}t; u) I_k(u)\} dW(u).$$

Remark. For the sake of simplicity let $N = 2$:

$$(5)\quad X(t) = Z_0(t) + \varphi(t) Z_1(t) = \int_0^t \{I_0(u) + \varphi(t) I(2^{-1}t; u) I_1(u)\} dW(u).$$

Relation (5) defines a linear transformation A of $\mathcal{H}(W;t)$ onto $\mathcal{H}(X;t)$ by $X(t) = AW(t)$. But A is unbounded. Indeed, $\|A(W(t+h) - W(t))\|^2 = \|X(t+h) - X(t)\|^2 = \|Z_1(t+h) - Z_1(t)\|^2 + \varphi^2(t+h)\|Z_2(t+h) - Z_2(t)\|^2 + [\varphi(t+h) - \varphi(t)]^2 \|Z_2(t)\|^2 = F_1(t+h) - F_1(t) + \varphi^2(t+h)[F_2(t+h) - F_2(t)]^2 + [\varphi(t+h) - \varphi(t)]^2 F_2(t)$, or, for $h \to 0$

$$\frac{\|A(W(t+h) - W(t))\|^2}{\|W(t+h) - W(t)\|^2} \longrightarrow F_1(t) + \varphi^2(t) F_2^1(t) + F_2(t) \lim_{h \to 0} \frac{[\varphi(t+h) - \varphi(t)]^2}{h}$$

The last limit is ∞ for some t. We would mention, in connection with the above remark, the hypothesis in [6], p.46. The simplified version of this hypothesis is the following: Let B be a regular linear transformation (i.e. B and B^{-1} are linear and bounded) defined by $Y(t) = BW(t)$. Then, the process $\{Y(t), t > 0\}$ has the spectral type equivalent to the spectral type of $\{W(t), t > 0\}$ i.e. the ordinary Lebesgue measure dt.

References

[1] Cramér, H.: Stochastic processes as curves in Hilbert Space, Teor. Veroya. i ee prim. 9(1964), 169-179.

[2] Ephremides, A. and Thomas, J.B.: On random processes linearly equivalent to white noise, Inform. Sci. 7(1974), 133-156.

[3] Hida, T.: Canonical representation of Gaussian processes and their applications, Mem. Coll. Sci., Univ. Kyoto, Ser. A, 33 (1960), 109-155.

[4] Pitt, L.D.: Hida-Cramer multiplicity theory for multiple Markov processes and Goursat representation, Nagoya Math.J. 57(1975), 199-228.

[5] Rozanov, Yu.A.: Innovation and non-anticipative processes, Multivariate Analysis-III Proceeding, Dayton, Ohio, (1972) New York, 1973.

[6] Rozanov, Yu.: Teoriya obnovlyayushchih procesov, Nauka, Moskva, 1974.

[7] Stoyanov, J.M.: Counterexamples in Probability, John Wiley and Sons, 1987.

REZIME

PROIZVOLJNOST MULTIPLICITETA ZA JEDNU LINEARNU NE - ANTICIPATIVNU TRANSFORMACIJU WIENEROVOG PROCESA

U radu je dat primer procesa $\{X(t), t > 0\}$ definisanog sa $X(t) = \int_0^t f(t;u)dW(u)$, ($\{W(t)\}$ je Vinerov proces) koji ima proizvoljni konačni multiplicitet N i apsolutno neprekidni maksimalni spektralni tip.

Received by the editors May 20, 1989.