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WEAK CONGRUENCES AND HOMOMORPHISMS

Branimir Šešelja and **Andreja Tepavčević**
Institute of Mathematics, University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

The necessary and sufficient conditions under which a weak congruence lattice of a factor algebra is a sublattice of a weak congruence lattice are given.

Some classes of algebras satisfying this condition are described (any variety of coherent algebras , ∪-algebras, etc.).

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1. Preliminaries

It is well known that for a congruence θ on an algebra $\mathcal{A}$, $Con\mathcal{A}/\theta$ is (up to the isomorphism) a sublattice $[\theta)$ of $Con\mathcal{A}$. For the weak congruence lattice $Cw\mathcal{A}$ (i.e. for the lattice of all congruences on all the subalgebras of $\mathcal{A}$, see for example [3]) it is not always the case. As an example, consider the four element groupoid $(G, *)$, given by its Cayley table.

*	a	b	c	d
a	a	c	b	b
b	c	b	d	d
c	b	d	a	a
d	b	d	a	a

$\mathcal{G} = (\{a, b, c, d\}, *)$, $\mathcal{A} = (\{a\}, *)$, $\mathcal{B} = (\{b\}, *)$
$Con\mathcal{G} = (\{\Delta, \theta, G^2\}, \leq)$, $\theta = \{\{a\}, \{b\}, \{c, d\}\}$,
$Sub\mathcal{G} = (\{\emptyset, A, B, G\}, \leq)$.

If θ is the congruence determined by the partition $\{\{a\}, \{b\}, \{c, d\}\}$, then the corresponding weak congruence lattice $Cw\mathcal{A}/\theta$ is not a sublattice of $Cw\mathcal{A}$ (see Fig.1). Note that $\mathcal{G}$ has two subgroupoids, both one-element.

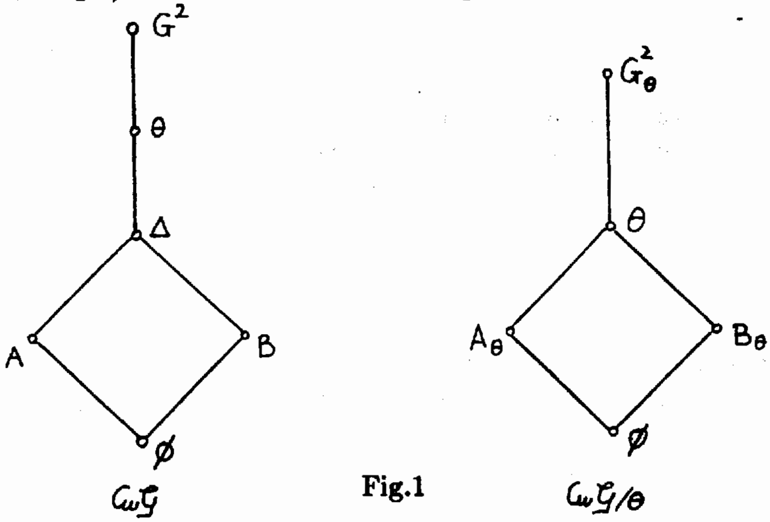

Fig.1

Algebras for which $Cw\mathcal{A}/\theta$ is a sublattice of $Cw\mathcal{A}$ ($\theta \in Con\mathcal{A}$) have to fulfill some lattice conditions (concerning the weak-congruence lattice), but also algebraic ones. To describe the former, we shall prove some pure lattice-theoretical propositions. For the latter, we shall use the well known facts about the construction of factor-algebras.

The notations, definitions, and some necessary results are from [2], [3], and from the references given there.

An element a of lattice L is **codistributive**, if for all $x, y \in L$

$$a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y),$$

or, equivalently, if and only if the mapping $m_a : x \to x \wedge a$ is a homomorphism of L into the ideal $(\ a]$.

A **distributive** element a of L is defined dually (the corresponding homomorphism from L into the filter $[\ a)$ is obvioisly $x \to x \vee a$).

An element a of L is said to be **modular**, if for all $x, y \in L$, $a \leq y$ implies $a \vee (x \wedge y) = (a \vee x) \wedge y$.

An element a of L is said to be **comodular**, if $a \leq y$ implies $x \vee (a \wedge y) = (x \vee a) \wedge y$.

The lattice $Cw\mathcal{A}$ of weak congruences of an algebra $\mathcal{A} = (A, F)$ is the lattice of all the symmetric and transitive subalgebras of $\mathcal{A}^2$ under the set inclusion. The diagonal relation $\Delta = \{(x,x) | \ x \in A\}$ is a codistributive element of that lattice, $[\ \Delta\ )$ is $Con\mathcal{A}$, and $Sub\mathcal{A}$ is isomorphic with $(\ \Delta]$ under $\rho \rightarrow \{x | x\rho x\}$.

Recall that $\mathcal{A}$ has the **congruence extension property** (CEP), if every congruence on a subalgebra of $\mathcal{A}$ is a restriction of the congruence on $\mathcal{A}$.

$\mathcal{A}$ has the CEP if and only if Δ is a comodular element of $Cw\mathcal{A}$ ([2]).

$\mathcal{A}$ is said to have the **congruence intersection property** (CIP), if for $\rho \in Con\mathcal{B}$, $\theta \in Con\mathcal{C}, \mathcal{B}, \mathcal{C} \in Sub\mathcal{A}$,

$$(\rho \cap \theta)_A = \rho_A \cap \theta_A,$$

where ρ_A stands for the least congruence on $\mathcal{A}$ extending ρ. Since $\rho_A = \rho \vee \Delta$ in $Cw\mathcal{A}$, $\mathcal{A}$ has the CIP if and only if Δ is a distributive element of $Cw\mathcal{A}$.

A is said to have the **weak congruence intersection property** (wCIP), if for $\rho \in Con\mathcal{B}$, $\theta \in Con\mathcal{A}$, $\mathcal{B} \in Sub\mathcal{A}$

$$(\rho \cap \theta)_A = \rho_A \cap \theta.$$

Obviously, $\mathcal{A}$ has the wCIP if and only if Δ is a modular element of $Cw\mathcal{A}$.

2. Sublattices determined by special elements

In the following, a will be a fixed codistributive element of the lattice L, such that the classes of the congruence induced by $m_a : x \rightarrow x \wedge a$ have maximum elements, the collection of which is denote by M_a. For $x \in L$, $\bar{x} \in M_a$, and $m_a(x) = m_a(\bar{x})$.

Since a is a codistributive element of L, we have that

$$L = \cup([a \wedge \bar{x},\ \bar{x}] |\ x \in L),$$

where the interval sublattice $[a \wedge \bar{x},\ \bar{x}]$ is a class of the congruence induced by m_a.

For $b \in [a)$, let $L_b = \cup([b \wedge \bar{x},\ \bar{x}] |\ x \in L)$. L_b is not necessarily a sublattice of L, and in the following proposition we give some necessary conditions under which it is.

Proposition 1. *Let a be a modular and comodular element of the lattice L, and let M_a be a sublattice of L. If b is a distributive element of the filter $[a)$, then L_b is a sublattice of L.*

Proof. Let $u, v \in \bigcup([b \wedge \bar{x},\ \bar{x}] \mid x \in L)$. Then $u \in [b \wedge \bar{y},\ \bar{y}]$, $v \in [b \wedge \bar{z},\ \bar{z}]$, for some $y, z \in L$, i.e. $b \wedge \bar{y} \leq u \leq \bar{y}$, and $b \wedge \bar{z} \leq v \leq \bar{z}$. Now

$$b \wedge \overline{y \wedge z} = b \wedge \bar{y} \wedge \bar{z} \leq u \wedge v \leq \bar{y} \wedge \bar{z} = \overline{y \wedge z},$$

and $u \wedge v \in [b \wedge \overline{y \wedge z}, \overline{y \wedge z}]$.

Moreover, $(b \wedge \bar{y}) \vee (b \wedge \bar{z}) \leq u \vee v \leq \bar{y} \wedge \bar{z} = \overline{y \vee z}$, since M_a is a sublattice of L.

By asumption a is modular in L and b distributive in $[\ a)$. Thus, $((b \wedge \bar{y}) \vee (b \wedge \bar{z})) \vee a = ((b \wedge \bar{y}) \vee a) \vee ((b \wedge \bar{z}) \vee a) = (b \wedge (\bar{y} \vee a)) \vee (b \wedge (\bar{z} \vee a)) = = b \wedge (\bar{y} \vee \bar{z} \vee a) = (b \wedge (\bar{y} \vee \bar{z})) \vee a$.

Now, since a is comodular, it follows that

$$(b \wedge \bar{y}) \vee (b \wedge \bar{z}) = b \wedge (\bar{y} \vee \bar{z}) = b \wedge (\overline{y \vee z}), \text{ and } u \vee v \in [b \wedge (\overline{y \vee z}), (\overline{y \vee z}].$$

□

Proposition 2. *Let M_a be a sublattice of L and take $b \in [a)$. If for $y, z \in L_b$*

$$b \wedge (y \vee z) = (b \wedge y) \vee (b \wedge z),$$

then L_b is a sublattice of L.

Proof. As in the previous proposition, if $u, v \in L_b$, then obviously $u \wedge v \in L_b$. If $b \wedge \bar{y} \leq u \leq \bar{y}$ and $b \wedge \bar{z} \leq \bar{z}$, then

$$b \wedge (\overline{y \vee z}) = b \wedge (\bar{y} \vee \bar{z}) = (b \wedge \bar{y}) \vee (b \wedge \bar{z}) \leq u \vee v \leq \overline{y \vee z},$$

which proves that L_b is a sublattice of L.

□

3. Weak congruence lattices of $\mathcal{A}$ and of $\mathcal{A}/\theta$

Let $\mathcal{A}$ be an algebra and $\theta \in Con\mathcal{A}$. If $\mathcal{B}_\theta \in Sub\mathcal{A}/\theta$, then $\mathcal{B} \in Sub\mathcal{A}$, and $B = B[\theta]$, where $B[\theta] = \{x \in A \mid x\theta b,$ for some $b \in B\}$.

For the congruence lattice of $\mathcal{B}/\theta$ we have that

$$Con\mathcal{B}/\theta \cong [B^2 \wedge \theta, B^2],$$

where $[B^2 \wedge \theta, B^2]$ is the interval sublattice of $Cw\mathcal{A}$. Consequently, $Cw\mathcal{A}/\theta$ is (up to the isomorphism) a sublattice of $Cw\mathcal{A}$, provided that

$$\cup([B^2 \wedge \theta, B^2] \mid \mathcal{B} \in Sub\mathcal{A},\ B = B[\theta])$$

is a sublattice of $Cw\mathcal{A}$.

Theorem 1. *Let $\mathcal{A}$ be an algebra, and $\theta \in Con\mathcal{A}$. Then, $Cw\mathcal{A}/\theta$ is a sublattice of $Cw\mathcal{A}$ if and only if the following conditions are satisfied for all $\mathcal{B},\mathcal{C} \in Sub\mathcal{A}$, such that $B = B[\theta]$, and $C = C[\theta]$:*

(i) $B \vee C = B \vee C[\theta]$;

(ii) $\theta \wedge (B^2 \vee C^2) = (\theta \wedge B^2) \vee (\theta \wedge C^2)$;

(iii) $\theta \wedge (B^2 \vee C^2) = \theta \wedge (B \vee C)^2$.

Proof. ($\Leftarrow$)

Let $B = B[\theta]$, and $C = C[\theta]$. If $\rho \in [B^2 \wedge \theta, B^2]$, $\phi \in [C^2 \wedge \theta, C^2]$, then $B^2 \wedge \theta \leq \rho \leq B^2$, $C^2 \wedge \theta \leq \phi \leq C^2$, and $(B \wedge C)^2 \wedge \theta \leq \rho \wedge \phi \leq B^2 \wedge C^2 = (B \wedge C)^2$. For the suprema, we have that $(B^2 \wedge \theta) \vee (C^2 \wedge \theta) \leq \rho \vee \phi$, and by (i), (ii). (iii) and since $B^2 \vee C^2 \leq (B \vee C)^2$, it follows that $(B \vee C)^2 \wedge \theta \leq \rho \vee \phi \leq (B \vee C)^2$, where $B \vee C = B \vee C[\theta]$.

($\Rightarrow$)

If $Cw\mathcal{A}/\theta$ is a sublattice of $Cw\mathcal{A}$, then from $B^2 \wedge \theta \leq \rho \leq B^2$ and $C^2 \wedge \theta \leq \phi \leq C^2$ for $B = B[\theta]$ and $C = C[\theta]$, it follows that $D^2 \wedge \theta \leq \rho \vee \phi \leq D^2$ where $\mathcal{D} \in Sub\mathcal{A}$, $D = D[\theta]$. Obviously, $\rho \vee \phi \in Con(\mathcal{B} \vee \mathcal{C})$, and thus $\mathcal{D} = \mathcal{B} \vee \mathcal{C}$. Hence, $B \vee C = B \vee C[\theta]$. Now, since $\rho = B^2 \wedge \theta$, $\phi = C^2 \wedge \theta$, it follows that

$$(B \vee C)^2 \wedge \theta \leq (B^2 \wedge \theta) \vee (C^2 \wedge \theta).$$

Finally, since

$$(\theta \wedge B^2) \vee (\theta \wedge C^2) \leq \theta \wedge (B^2 \vee C^2) \leq \theta \wedge (B \vee C)^2,$$

the proof is complete.

□

It is obvious that the collection

$$\{B^2 \wedge \theta \mid \mathcal{B} \in Sub\mathcal{A},\ B = B[\theta]\}$$

represents, (up to the isomorphism) all the subalgebras of $\mathcal{A}/\theta$ ($\theta \in Con\mathcal{A}$). Hence, the previous proposition can be formulated in the following way.

Corollary 1. *For an algebra $\mathcal{A}$, and $\theta \in Con\mathcal{A}$, $Cw\mathcal{A}/\theta$ is a sublattice of $Cw\mathcal{A}$ if and only if $Sub\mathcal{A}/\theta$ is a sublattice of $Cw\mathcal{A}$.*

Proposition 3. *Let $\mathcal{A}$ be an algebra for which $B_m \neq \emptyset$, where*

$$B_m = \cap(B \mid \mathcal{B} \in Sub\mathcal{A}),$$

and with the property that for $\mathcal{B},\mathcal{C} \in Sub\mathcal{A}$, if $B = B[\theta]$ and $C = C[\theta]$, then $B \vee C = B \vee C[\theta]$. Now, if θ is a codistributive element of $Cw\mathcal{A}$, then $Cw\mathcal{A}/\theta$ is a sublattice of $Cw\mathcal{A}$.

Proof. Immediately by Proposition 2.

□

Proposition 4. *Let $\mathcal{A}$ be such that $B_m \neq \emptyset$, and that for $\mathcal{B},\mathcal{C} \in Sub\mathcal{A}$, $B = B[\theta]$, $C = C[\theta]$, $B \vee C = B \vee C[\theta]$. If $\mathcal{A}$ satisfies the CEP and the wCIP, and if θ is a codistributive element of $Con\mathcal{A}$,then $Cw\mathcal{A}/\theta$ is a sublattice of $Cw\mathcal{A}$.*

Proof. By Proposition 1. and by Theorem 1.

□

Proposition 4 gives some sufficient conditions under which $Cw\mathcal{A}/\theta$ is a sublattice of $Cw\mathcal{A}$ (for $\theta \in Con\mathcal{A}$). However,these conditions are not necessary. In the following example ([3]), $Cw\mathcal{A}/\theta$ is a sublattice of $Cw\mathcal{A}$, but the groupoid $\mathcal{A}$ does not satisfy any of the required conditions (namely, $\mathcal{A}$ has neither the CEP, nor the wCIP, and $B_m = \emptyset$, see Fig.2 and 3).

$$\mathcal{A} = (\{a,b,c,d\},*)$$

*	a	b	c	d
a	c	c	c	c
b	c	c	c	d
c	c	c	b	d
d	d	d	d	d

$\theta : \{\{a\},\{b,c\},\{d\}\}$

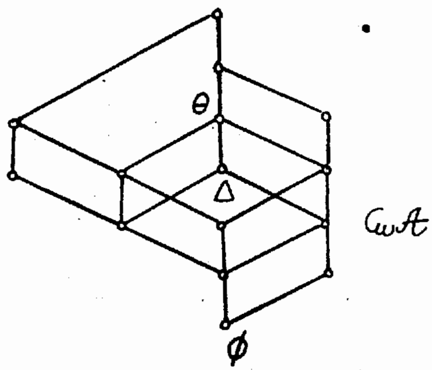

Fig.2

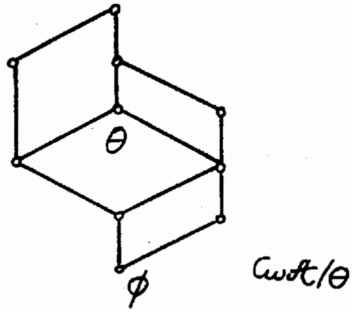

Fig.3

Which known classes of algebras have a property that $Cw\mathcal{A}/\theta$ is a sublattice of $Cw\mathcal{A}$?

Recall that a **coherent** algebra is the one having the property that if any subalgebra of it contains a class of some congruence, then that subalgebra is a union of classes of that congruence. Obviously, for a coherent algebra $\mathcal{A}$, if $\mathcal{B},\mathcal{C} \in Sub\mathcal{A}$ and $B = B[\theta]$, $C = C[\theta]$, then $B \vee C = B \vee C[\theta]$.

An algebra $\mathcal{A}$ is **regular** if every congruence on $\mathcal{A}$ is uniquely determined by any of its classes.

Theorem 2. *If $\mathcal{A}$ is a coherent algebra and if all its subalgebras are regular, then, for any $\theta \in Con\mathcal{A}$, $Cw\mathcal{A}/\theta$ is a sublattice of $Cw\mathcal{A}$.*

Proof. Since $\mathcal{A}$ is coherent, all we have to prove is that for $\theta \in Con\mathcal{A}$, and for $\mathcal{B},\mathcal{C} \in Sub\mathcal{A}$, $B = B[\theta]$, $C = C[\theta]$,

$$\theta \wedge (B^2 \vee C^2) = (\theta \wedge B^2) \vee (\theta \wedge C^2).$$

Let $\phi = \theta \wedge (B \vee C)^2$. Obviously, $\phi \in Con(\mathcal{B} \vee \mathcal{C})$, and ϕ is a union of the classes of θ. On the other hand, the congruence $\psi = (\theta \wedge B^2) \vee (\theta \wedge C^2)$ also belongs to $Con(\mathcal{B} \vee \mathcal{C})$, and contains the union of the classes from $\theta \wedge B^2$ and $\theta \wedge C^2$. The same classes are contained in ϕ. But $\psi \leq \phi$, and since $\mathcal{B} \vee \mathcal{C}$ is regular, $\psi = \phi$.

□

Corollary 2. *For every group G, and $N \triangleleft G$, $C\omega G/N$ is a sublattice of $C\omega G$.*

Proof. Obvious, since every group is coherent and regular.

□

Corollary 3. *If $\mathcal{A}$ is an algebra in a coherent variety, then for every $\theta \in Con\mathcal{A}$ $Cw\mathcal{A}/\theta$ is a sublattice of $Cw\mathcal{A}$.*

Proof.. It is known ([1]) that any coherent variety is congruence regular.

□

Not only coherent algebras have the required property.

Let $\mathcal{A}$ be a $\cup$-algebra, i.e. the algebra for which the set-theoretic union of any two subalgebras is again a subalgebra of $\mathcal{A}$.

Theorem 3. *If $\mathcal{A}$ is a $\cup$-algebra, then for $\theta \in Con\mathcal{A}$, $Cw\mathcal{A}/\theta$ is a sublattice of $Cw\mathcal{A}$.*

Proof. If $\theta \in Con\mathcal{A}$, $\mathcal{B},\mathcal{C} \in Sub\mathcal{A}$, $B = B[\theta]$ and $C = C[\theta]$, then

$$B \vee C = B \cup C = B \vee C[\theta].$$

Moreover, if $[x]_\theta$ is a θ-congruence class to which $x \in A$ belongs, then

$$\cup([x]_\theta \mid x \in B \cup C) = \cup([x]_\theta \mid x \in B) \cup \cup([x]_\theta \mid x \in C) \text{i.e.}$$

$$\theta \cap (B \cup C)^2 = (\theta \cap B^2) \cup (\theta \cap C^2).$$

Now, $\theta \cap (B \cup C)^2 \in Con(B \cup C)$ and thus

$$(\theta \cap B^2) \cup (\theta \cap C^2) = (\theta \wedge B^2) \vee (\theta \wedge C^2)$$

in $Cw\mathcal{A}$. Hence,

$$(\theta \wedge (B \vee C)^2) = (\theta \wedge B^2) \vee (\theta \wedge C^2),$$

and the proof is complete by Theorem 1.

□

Corollary 4. *If $\mathcal{A}$ is a unary algebra, then for $\theta \in Con\mathcal{A}$, $Cw\mathcal{A}/\theta$ is a sublattice of $Cw\mathcal{A}$.*

Proof. Obvious, since $\mathcal{A}$ is a $\cup$ -algebra.

□

References

[1] Geiger, D.: Congruence coherent algebras, Abstract 74T-A130, Notices Amer. Math. Soc. 21(1974) A-436.

[2] Šešelja, B., Tepavčević, A.: On CEP and semimodularity in the lattice of weak congruences, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. (to appear).

[3] Vojvodić, G., Šešelja, B.: On the lattice of weak congruence relations, Algebra Universalis, 25(1988), 121–130.

REZIME

SLABE KONGRUENCIJE I HOMOMORFIZMI

U radu se daju potrebni i dovoljni uslovi u odnosu na koje je mreža slabih kongruencija homomorfne slike neke algebre podmreža (do na izomorfizam) mreže slabih kongruencija same algebre. Opisane su neke klase algebri koje taj uslov zadovoljavaju (varijeteti koherentnih algebri, U-algebre itd.).

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